DUALITY WITH
EXPANDING MAPS AND SHRINKING MAPS, AND
ITS APPLICATIONS TO GAUSS MAPS

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ABSTRACT. We study expanding maps and shrinking maps of subvarieties of Grassmann varieties in arbitrary characteristic. The shrinking map was studied independently by Landsberg and Piontkowski in order to characterize Gauss images. To develop their method, we introduce the expanding map, which is a dual notion of the shrinking map and is a generalization of the Gauss map. Then we give a characterization of separable Gauss maps and their images, which yields results for the following topics: (1) Linearity of general fibers of separable Gauss maps; (2) Generalization of the characterization of Gauss images; (3) Duality on one-dimensional parameter spaces of linear subvarieties lying in developable varieties.

1. INTRODUCTION

For a projective variety $X \subset \mathbb{P}^N$ over an algebraically closed field of arbitrary characteristic, the *Gauss map* $\gamma = \gamma_X$ of X is defined to be the rational map $X \dashrightarrow \mathbb{G}(\dim X, \mathbb{P}^N)$ which sends each smooth point x to the embedded tangent space $\mathbb{T}_x X$ at x in $\mathbb{P}^N$. The *shrinking map* of a subvariety of a Grassmann variety was studied independently by Landsberg and Piontkowski in order to characterize *Gauss images* in characteristic zero, around 1996 according to [11, p. 93] (see [1, 2.4.7] and [11, Theorem 3.4.8] for details of their results). To develop their method, we introduce the *expanding map* of a subvariety of a Grassmann variety, which is a generalization of the Gauss map and is a dual notion of the shrinking map (see §2 for precise definitions of these maps). Then we have the main theorem, Theorem 3.1, which is a characterization of separable Gauss maps and their images in arbitrary characteristic, and which yields results for the following topics.

1.1. **Linearity of general fibers of separable Gauss maps.**

Theorem 1.1 (= Corollary 3.6)**.** *Let γ be a separable Gauss map of a projective variety $X \subset \mathbb{P}^N$. Then the closure of a general fiber of γ is a linear subvariety of $\mathbb{P}^N$* [1].

[1] In the first version of this paper (arXiv:1110.4841v1), we proved the linearity by using duality on subspaces of tangent spaces with linear projection techniques. After that, we have

According to a theorem of Zak [26, I, 2.8. Corollary], the Gauss map is finite if X is smooth (and is not a linear subvariety of $\mathbb{P}^N$). Combining with Theorem 1.1, we have that, if the projective variety $X \subset \mathbb{P}^N$ is smooth and the Gauss map γ is separable, then γ is in fact birational (Corollary 3.7). Geometrically, the birationality of γ means that a general embedded tangent space is tangent to X at a unique point.

In characteristic zero, it was well known that the closure $F \subset X$ of a general fiber of the Gauss map γ is a linear subvariety of $\mathbb{P}^N$ (Griffiths and Harris [7, (2.10)], Zak [26, I, 2.3. Theorem (c)]). In positive characteristic, γ can be *inseparable*, and then F can be *non-linear* (see Remark 3.8); this leads us to a natural question: Is F a linear subvariety if γ is *separable*? (Kaji asked, for example, in [16, Question 2] [17, Problem 3.11].) The curve case was classically known (see Remark 3.9). Kleiman and Piene [20, pp. 108–109] proved that, if $X \subset \mathbb{P}^N$ is *reflexive*, then a general fiber of the Gauss map γ is scheme-theoretically (an open subscheme of) a linear subvariety of $\mathbb{P}^N$. In characteristic zero, their result gives a reasonable proof of the linearity of F, since every X is reflexive. In arbitrary characteristic, in terms of reflexivity, the linearity of F of a separable γ follows if $\mathrm{codim}_{\mathbb{P}^N}(X) = 1$ or $\dim X \leqslant 2$, since separability of γ implies reflexivity of X if $\mathrm{codim}_{\mathbb{P}^N}(X) = 1$ (due to the Monge-Segre-Wallace criterion [8, (2.4)], [19, I-1(4)]), $\dim X = 1$ (Voloch [24], Kaji [14]), or $\dim X = 2$ (Fukasawa and Kaji [6]). On the other hand, for $\dim X \geqslant 3$, Kaji [15] and Fukasawa [4] [5] showed that separability of γ does *not* imply reflexivity of X in general. For any X, by Theorem 1.1, we finally answer the question affirmatively.

1.2. **Generalization of the characterization of Gauss images.** We generalize the characterization of Gauss images given by Landsberg and Piontkowski to the arbitrary characteristic case, as follows:

Theorem 1.2 (= Corollary 3.14)**.** *Let σ be the shrinking map from a closed subvariety $\mathcal{Y} \subset \mathbb{G}(M, \mathbb{P}^N)$ to $\mathbb{G}(M^-, \mathbb{P}^N)$ with integers M, M^- ($M \geqslant M^-$), and let $U_{\mathbb{G}(M^-, \mathbb{P}^N)} \subset \mathbb{G}(M^-, \mathbb{P}^N) \times \mathbb{P}^N$ be the universal family of $\mathbb{G}(M^-, \mathbb{P}^N)$. Then $\mathcal{Y}$ is the closure of a image of a separable Gauss map if and only if $M^- = M - \dim \mathcal{Y}$ holds and the projection $\sigma^* U_{\mathbb{G}(M^-, \mathbb{P}^N)} \to \mathbb{P}^N$ is separable and generically finite onto its image.*

Here the *generalized conormal morphism*, induced from a expanding map, plays an essential role; indeed, we give a generalization of the Monge-Segre-Wallace criterion to the morphism (Proposition 3.12).

1.3. **Duality on one-dimensional developable parameter spaces.** Later in the paper, instead of the subvariety $X \subset \mathbb{P}^N$, we focus on $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$, a parameter space of m-planes lying in X, and study *developability* of $(\mathcal{X}, X)$ (see §4 for definitions). It is classically known that, in characterize zero, a projective

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examined what is essential in the proof. Now, in this second version, considering expanding maps and shrinking maps, we give a more evident proof of the result.

variety having a one-parameter developable uniruling (by m-planes) is obtained as a cone over an *osculating scroll* of a curve ([1, 2.2.8], [11, Theorem. 3.12.5]; the arbitrary characteristic case was investigated by Fukasawa [2]). Applying our main theorem, we find duality on one-dimensional developable parameter spaces via expanding maps and shrinking maps, in arbitrary characteristic, as follows. Here $\gamma^i = \gamma^i_{\mathcal{X}}$ is defined inductively by $\gamma^1 := \gamma, \gamma^i := \gamma_{\gamma^{i-1}\mathcal{X}} \circ \gamma^{i-1}$, with the closure $\gamma^i \mathcal{X}$ of the image of $\mathcal{X}$ under γ^i. In a similar way, σ^i is defined.

Theorem 1.3 (= Theorem 4.15). *Let $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$ and $\mathcal{X}' \subset \mathbb{G}(m', \mathbb{P}^N)$ be projective curves. Then, for an integer $\varepsilon \geqslant 0$, the following are equivalent:*

(a) *$\mathcal{X}'$ is developable, the map $\gamma^\varepsilon = \gamma^\varepsilon_{\mathcal{X}'}$ is separable and generically finite, and $\gamma^\varepsilon \mathcal{X}' = \mathcal{X}$.*

(b) *$\mathcal{X}$ is developable, the map $\sigma^\varepsilon = \sigma^\varepsilon_{\mathcal{X}}$ is separable and generically finite, and $\sigma^\varepsilon \mathcal{X} = \mathcal{X}'$.*

In this case, $m = m' + \varepsilon$.

As a corollary, if σ^m is separable and X is not a cone, then $C := \sigma^m \mathcal{X}$ is a projective curve in $\mathbb{P}^N$ such that γ^m is separable and $\mathcal{X} = \gamma^m C$; in particular, X is equal to the osculating scroll of order m of C (Corollary 4.16). On the other hand, if $\gamma^2_{\mathcal{X}}$ is separable, then an equality $T((TX)^*) = X^*$ in $(\mathbb{P}^N)^\vee$ holds (Corollary 4.17; cf. for osculating scrolls of curves, this equality was deduced from Piene's work in characterization zero [22], and was shown by Homma under some conditions on the characteristic [9] (see [9, Remark 4.3])).

This paper is organized as follows: In §2 we fix our notation and give a local parametrization of a expanding map $\gamma : \mathcal{X} \dashrightarrow \mathbb{G}(m^+, \mathbb{P}^N)$ of a subvariety $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$. In addition, setting $\mathcal{Y}$ to be the closure of the image of $\mathcal{X}$, we investigate properties of composition of the expanding map γ of $\mathcal{X}$ and the shrinking map σ of $\mathcal{Y}$. Then, in §3 we prove the main theorem, Theorem 3.1. In §4 we regard $\mathcal{X}$ as a parameter space of m-planes lying in $X \subset \mathbb{P}^N$, and study developability of $\mathcal{X}$ in terms of γ.

2. Expanding maps of subvarieties of Grassmann varieties

In this section, we denote by $\gamma : \mathcal{X} \dashrightarrow \mathbb{G}(m^+, \mathbb{P}^N)$ the expanding map of a subvariety $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$ with integer m, m^+ ($m \leqslant m^+$), which is defined as follows:

Definition 2.1. Let $\mathcal{Q}_{\mathbb{G}(m, \mathbb{P}^N)}$ and $\mathcal{S}_{\mathbb{G}(m, \mathbb{P}^N)}$ be the universal quotient bundle and subbundle of rank $m + 1$ and $N - m$ on $\mathbb{G}(m, \mathbb{P}^N)$ with the exact sequence $0 \to \mathcal{S}_{\mathbb{G}(m, \mathbb{P}^N)} \to H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{G}(m, \mathbb{P}^N)} \to \mathcal{Q}_{\mathbb{G}(m, \mathbb{P}^N)} \to 0$. We set $\mathcal{Q}_{\mathcal{X}} := \mathcal{Q}_{\mathbb{G}(m, \mathbb{P}^N)}|_{\mathcal{X}}$ and call this the universal quotient bundle on $\mathcal{X}$, and so on. A homomorphism φ is defined by the composition:

$$\varphi : \mathcal{S}_{\mathcal{X}^{sm}} \to \mathcal{H}om(\mathcal{H}om(\mathcal{S}_{\mathcal{X}^{sm}}, \mathcal{Q}_{\mathcal{X}^{sm}}), \mathcal{Q}_{\mathcal{X}^{sm}}) \to \mathcal{H}om(T_{\mathcal{X}^{sm}}, \mathcal{Q}_{\mathcal{X}^{sm}}),$$

where the first homomorphism is induced from the dual of $\mathcal{Q}_{\mathcal{X}} \otimes \mathcal{Q}_{\mathcal{X}}^\vee \to \mathcal{O}$, and the second one is induced from $T_{\mathcal{X}^{sm}} \hookrightarrow T_{\mathbb{G}(m, \mathbb{P}^N)}|_{\mathcal{X}^{sm}} = \mathcal{H}om(\mathcal{S}_{\mathcal{X}^{sm}}, \mathcal{Q}_{\mathcal{X}^{sm}})$. We

can take an integer $m^+ = m_\gamma^+$ with $m \leqslant m^+ \leqslant N$ such that a general point $x \in \mathcal{X}$ satisfies

$$\dim(\ker \varphi \otimes k(x)) = N - m^+.$$

Let $(\mathbb{P}^N)^\vee := \mathbb{G}(N-1, \mathbb{P}^N)$, the space of hyperplanes. Then $\ker \varphi|_{\mathcal{X}^\circ}$ is a subbundle of $H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathcal{X}^\circ} \simeq H^0((\mathbb{P}^N)^\vee, \mathcal{O}(1))^\vee \otimes \mathcal{O}_{\mathcal{X}^\circ}$ of rank $N - m^+$ for a certain open subset $\mathcal{X}^\circ \subset \mathcal{X}$. By the universality of the Grassmann variety, under the identification $\mathbb{G}(N - m^+ - 1, (\mathbb{P}^N)^\vee) \simeq \mathbb{G}(m^+, \mathbb{P}^N)$, we have an induced morphism,

$$\gamma = \gamma_{\mathcal{X}/\mathbb{G}(m,\mathbb{P}^N)} : \mathcal{X}^\circ \to \mathbb{G}(m^+, \mathbb{P}^N).$$

We call γ the *expanding map* of $\mathcal{X}$. Here $\ker \varphi|_{\mathcal{X}^\circ} \simeq \gamma|_{\mathcal{X}^\circ}^*(\mathcal{S}_{\mathbb{G}(m^+,\mathbb{P}^N)})$.

Remark 2.2. Suppose that $m = 0$ and $X \subset \mathbb{P}^N = \mathbb{G}(0, \mathbb{P}^N)$. Then $\gamma = \gamma_{X/\mathbb{P}^N}$ coincides with the Gauss map $X \dashrightarrow \mathbb{G}(\dim(X), \mathbb{P}^N)$; in other words, $\gamma(x) = \mathbb{T}_x X$ for each smooth point $x \in X$. The reason is as follows: In this setting, it follows that $\mathcal{S}_{\mathbb{P}^N} = \Omega_{\mathbb{P}^N}^1(1)$ and $\mathcal{Q}_{\mathbb{P}^N} = \mathcal{O}_{\mathbb{P}^N}(1)$, and that φ is the homomorphism $\Omega_{\mathbb{P}^N}^1(1)|_X \to \Omega_X^1(1)$. Therefore $\ker \varphi|_{\mathcal{X}^{sm}} = N_{X/\mathbb{P}^N}^\vee(1)|_{\mathcal{X}^{sm}}$, which implies the assertion.

The shrinking map $\sigma : \mathcal{Y} \dashrightarrow \mathbb{G}(M^-, \mathbb{P}^N)$ of a subvariety $\mathcal{Y} \subset \mathbb{G}(M, \mathbb{P}^N)$ with integers M, M^- $(M \geqslant M^-)$ is defined similarly, as follows:

Definition 2.3. Let $\mathcal{Q}_\mathcal{Y}$ and $\mathcal{S}_\mathcal{Y}$ be the universal quotient bundle and subbundle of rank $M + 1$ and $N - M$ on $\mathcal{Y}$. A homomorphism Φ is defined by the composition:

$$\Phi : \mathcal{Q}_{\mathcal{Y}^{sm}}^\vee \to \mathcal{H}om(\mathcal{H}om(\mathcal{Q}_{\mathcal{Y}^{sm}}^\vee, \mathcal{S}_{\mathcal{Y}^{sm}}^\vee), \mathcal{S}_{\mathcal{Y}^{sm}}^\vee) \to \mathcal{H}om(T_{\mathcal{Y}^{sm}}, \mathcal{S}_{\mathcal{Y}^{sm}}^\vee),$$

where the second homomorphism is induced from $T_{\mathcal{Y}^{sm}} \hookrightarrow T_{\mathbb{G}(M,\mathbb{P}^N)}|_{\mathcal{Y}^{sm}} = \mathcal{H}om(\mathcal{Q}_{\mathcal{Y}^{sm}}^\vee, \mathcal{S}_{\mathcal{Y}^{sm}}^\vee)$. We can take an integer $M^- = M_\sigma^-$ with $-1 \leqslant M^- \leqslant M$ such that a general point $y \in \mathcal{Y}$ satisfies

$$\dim(\ker \Phi \otimes k(y)) = M^- + 1.$$

Since $\ker \Phi|_{\mathcal{Y}^\circ}$ is a subbundle of $H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee \otimes \mathcal{O}_{\mathcal{Y}^\circ}$ of rank $M^- + 1$ for a certain open subset $\mathcal{Y}^\circ \subset \mathcal{Y}$, we have an induced morphism, called the *shrinking map* of $\mathcal{Y}$,

$$\sigma = \sigma_{\mathcal{Y}/\mathbb{G}(M,\mathbb{P}^N)} : \mathcal{Y}^\circ \to \mathbb{G}(M^-, \mathbb{P}^N).$$

Here we have $\ker \Phi|_{\mathcal{Y}^\circ} = \sigma|_{\mathcal{Y}^\circ}^*(\mathcal{Q}_{\mathbb{G}(M^-,\mathbb{P}^N)}^\vee)$.

Remark 2.4. Let $\bar{\mathcal{X}} \subset \mathbb{G}(N-m-1, (\mathbb{P}^N)^\vee)$ be the subvariety corresponding to $\mathcal{X}$ under the identification $\mathbb{G}(m, \mathbb{P}^N) \simeq \mathbb{G}(N-m-1, (\mathbb{P}^N)^\vee)$, and so on. Then $\gamma_{\mathcal{X}/\mathbb{G}(m,\mathbb{P}^N)}$ is identified with the shrinking map

$$\sigma_{\bar{\mathcal{X}}/\mathbb{G}(N-m-1,(\mathbb{P}^N)^\vee)} : \bar{\mathcal{X}} \dashrightarrow \mathbb{G}(N-m^+-1, (\mathbb{P}^N)^\vee)$$

under $\mathbb{G}(m^+, \mathbb{P}^N) \simeq \mathbb{G}(N-m^+-1, (\mathbb{P}^N)^\vee)$. In a similar way, $\sigma_{\mathcal{Y}/\mathbb{G}(M,\mathbb{P}^N)}$ is identified with the expanding map $\gamma_{\bar{\mathcal{Y}}/\mathbb{G}(N-M-1,(\mathbb{P}^N)^\vee)}$.

Let $U_{\mathbb{G}(m,\mathbb{P}^N)} \subset \mathbb{G}(m,\mathbb{P}^N) \times \mathbb{P}^N$ be the universal family of $\mathbb{G}(m,\mathbb{P}^N)$. We denote by $U_{\mathcal{X}} := U_{\mathbb{G}(m,\mathbb{P}^N)}|_{\mathcal{X}} \subset \mathcal{X} \times \mathbb{P}^N$ the universal family of $\mathcal{X}$, and by $\pi_{\mathcal{X}} : U_{\mathcal{X}} \to \mathbb{P}^N$ the projection, and so on. (Recall that, for each $x \in \mathcal{X}$, the m-plane $x \subset \mathbb{P}^N$ is equal to $\pi_{\mathcal{X}}(L_x)$ for the fiber L_x of $U_{\mathcal{X}} \to \mathcal{X}$ at x.)

Remark 2.5. A general point $x \in \mathcal{X}$ gives an inclusion $x \subset \gamma(x)$ of linear varieties in $\mathbb{P}^N$, and a general point $y \in \mathcal{Y}$ gives an inclusion $\sigma(y) \subset y$ of linear varieties in $\mathbb{P}^N$.

Lemma 2.6. *Let $\mathcal{X}, \mathcal{Y}$ be as above. Then the following holds:*

(a) *If $m^+ \leqslant N - 1$ and the image of γ is a point, then $\pi_{\mathcal{X}}(U_{\mathcal{X}}) \subset \mathbb{P}^N$ is degenerate.*

(b) *If $M^- \geqslant 0$ and the image of σ is a point, then $\pi_{\mathcal{Y}}(U_{\mathcal{Y}}) \subset \mathbb{P}^N$ is a cone.*

Proof. (a) Assume that the image of γ is a point $L \in \mathbb{G}(m^+, \mathbb{P}^N)$. For general $x \in \mathcal{X}$, we have $x \subset \gamma(x) = L$ as in Remark 2.5. Thus $\pi_{\mathcal{X}}(U_{\mathcal{X}})$ is contained in the m^+-plane L, and hence is degenerate.

(b) Assume that the image of σ is a point $L \in \mathbb{G}(M^-, \mathbb{P}^N)$, and take a point $v \in L \subset \mathbb{P}^N$. For general $y \in \mathcal{Y}$, we have $\sigma(y) = L \subset y$, which implies $v \in y$. We set $Y' := \pi_{\mathcal{Y}}(\mathcal{Y}) \subset \mathbb{P}^N$. Then a general point $y' \in Y'$ is contained in some M-plane y, where the line $\overline{y'v}$ is contained in $y \subset Y'$. Hence Y' is a cone with vertex v. $\qquad\square$

We denote by $\mathbb{P}_*(A) := \mathrm{Proj}(\bigoplus \mathrm{Sym}^d A^\vee)$ the projectivization of a locally free sheaf or a vector space A.

Definition 2.7. Let

$$V_{\mathbb{G}(M,\mathbb{P}^N)} := \mathbb{P}_*(\mathcal{S}_{\mathbb{G}(M,\mathbb{P}^N)}),$$

which is contained in $\mathbb{G}(M,\mathbb{P}^N) \times (\mathbb{P}^N)^\vee = \mathbb{P}_*(H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{G}(M,\mathbb{P}^N)})$ and is regarded as the universal family of $\mathbb{G}(N - M - 1, (\mathbb{P}^N)^\vee)$. We set $V_{\mathcal{Y}} := V_{\mathbb{G}(M,\mathbb{P}^N)}|_{\mathcal{Y}}$ and set $\bar{\pi} = \bar{\pi}_{\mathcal{Y}} : V_{\mathcal{Y}} \to (\mathbb{P}^N)^\vee$ to be the projection.

In the case where $\mathcal{Y}$ is the closure of the image of $\mathcal{X}$ under the expanding map γ, the following commutative diagram is obtained:

$$
\begin{array}{ccc}
& \gamma^*\bar{\pi} & \\
\gamma^*V_{\mathcal{Y}} \dashrightarrow V_{\mathcal{Y}} \xrightarrow{\ \bar{\pi}\ } & (\mathbb{P}^N)^\vee \\
\downarrow \qquad\qquad \downarrow & \\
\mathcal{X} \dashrightarrow[\gamma] \mathcal{Y}, &
\end{array}
$$

where we call the projection $\gamma^*\bar{\pi} : \gamma^*V_{\mathcal{Y}} \to (\mathbb{P}^N)^\vee$ the *generalized conormal morphism*, and where $\gamma^*V_{\mathcal{Y}} \subset \mathcal{X} \times (\mathbb{P}^N)^\vee$ is the closure of the pull-back $(\gamma|_{\mathcal{X}^\circ})^*V_{\mathcal{Y}}$. Note that we have $(\gamma|_{\mathcal{X}^\circ})^*V_{\mathcal{Y}} = \mathbb{P}_*(\ker \varphi|_{\mathcal{X}^\circ})$, because of $\gamma|_{\mathcal{X}^\circ}^*(\mathcal{S}_{\mathcal{Y}}) \simeq \ker \varphi|_{\mathcal{X}^\circ}$.

2.1. **Standard open subset of the Grassmann variety.** Let us denote by

$$(Z^0 : Z^1 : \cdots : Z^N)$$

the homogeneous coordinates on $\mathbb{P}^N$. To fix our notation, we will prepare a description of a standard open subset $\mathbb{G}_m^\circ \subset \mathbb{G}(m, \mathbb{P}^N)$ which is the set of m-planes not intersecting the $(N-m-1)$-plane $(Z^0 = Z^1 = \cdots = Z^m = 0)$. Let us denote by

$$Z_0, Z_1, \ldots, Z_N \in H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee$$

the dual basis of $Z^0, Z^1, \ldots, Z^N \in H^0(\mathbb{P}^N, \mathcal{O}(1))$, and so on.

(A) The sheaves $\mathcal{Q}_{\mathbb{G}_m^\circ}$ and $\mathcal{S}_{\mathbb{G}_m^\circ}^\vee$ are free on $\mathbb{G}_m^\circ$, and are equal to $Q \otimes \mathcal{O}_{\mathbb{G}_m^\circ}$ and $S^\vee \otimes \mathcal{O}_{\mathbb{G}_m^\circ}$, for the vector spaces

$$Q := \bigoplus_{0 \leqslant i \leqslant m} K \cdot \eta^i \ \text{ and } \ S^\vee := \bigoplus_{m+1 \leqslant j \leqslant N} K \cdot \zeta_j,$$

where K is the ground field, η^i is the image of Z^i under $H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O} \to \mathcal{Q}_{\mathbb{G}(m,\mathbb{P}^N)}$, and ζ_j is the image of Z_j under $H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee \otimes \mathcal{O} \to \mathcal{S}_{\mathbb{G}(m,\mathbb{P}^N)}^\vee$.

We have a standard isomorphism

$$(1) \qquad \mathbb{G}_m^\circ \simeq \mathrm{Hom}(Q^\vee, S^\vee) : x \mapsto \sum_{0 \leqslant i \leqslant m, m+1 \leqslant j \leqslant N} a_i^j \cdot \eta^i \otimes \zeta_j = (a_i^j)_{i,j},$$

as follows. We take an element $x \in \mathbb{G}_m^\circ$. Under the surjection $H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee \to \mathcal{S}_{\mathbb{G}(m,\mathbb{P}^N)}^\vee \otimes k(x)$, for each $0 \leqslant i \leqslant m$, we have $Z_i \mapsto -\sum_{m+1 \leqslant j \leqslant N} a_i^j \cdot \zeta_j$ with some $a_i^j = a_i^j(x) \in K$. This induces a linear map $Q^\vee \to S^\vee : \eta_i \mapsto \sum a_i^j \cdot \zeta_j$, which is regarded as a tensor $\sum a_i^j \cdot \eta^i \otimes \zeta_j$ under the identification $\mathrm{Hom}(Q^\vee, S^\vee) \simeq Q \otimes S^\vee$. This gives the homomorphism (1).

In this setting, the linear map $\mathcal{Q}_{\mathbb{G}_m^\circ}^\vee \otimes k(x) \to H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee$ is given by $\eta_i \mapsto Z_i + \sum_j a_i^j \cdot Z_j$, and hence, for each $x \in \mathbb{G}(m, \mathbb{P}^N)$, the m-plane $x \subset \mathbb{P}^N$ is spanned by the points of $\mathbb{P}^N$ corresponding to the row vectors of the $(m+1) \times (N+1)$ matrix,

$$\begin{bmatrix} 1 & & & \mathbf{0} & a_0^{m+1} & \cdots & a_0^N \\ & 1 & & & a_1^{m+1} & \cdots & a_1^N \\ & & \ddots & & \vdots & & \vdots \\ \mathbf{0} & & & 1 & a_m^{m+1} & \cdots & a_m^N \end{bmatrix}.$$

(B) Let $U_{\mathbb{G}(m,\mathbb{P}^N)} := \mathbb{P}_*(\mathcal{Q}_{\mathbb{G}(m,\mathbb{P}^N)}^\vee)$ in $\mathbb{G}(m, \mathbb{P}^N) \times \mathbb{P}^N$, which is the universal family of $\mathbb{G}(m, \mathbb{P}^N)$. Then we have an identification

$$U_{\mathbb{G}_m^\circ} \simeq \mathbb{G}_m^\circ \times \mathbb{P}^m \simeq \mathrm{Hom}(Q^\vee, S^\vee) \times \mathbb{P}^m.$$

Regarding $(\eta^0 : \cdots : \eta^m)$ as the homogeneous coordinates on $\mathbb{P}^m = \mathbb{P}_*(Q^\vee)$, under the identification (1), we can parametrize the projection $U_{\mathbb{G}_m^\circ} \to \mathbb{P}^N$ by sending $((a_i^j)_{i,j}, (\eta^0 : \cdots : \eta^m))$ to the point

$$(\eta^0 : \cdots : \eta^m : \sum_i \eta^i a_i^{m+1} : \cdots : \sum_i \eta^i a_i^N) \in \mathbb{P}^N.$$

This is also expressed as

$$(2) \qquad \sum_{0 \leqslant i \leqslant m} \eta^i Z_i + \sum_{0 \leqslant i \leqslant m, m+1 \leqslant j \leqslant N} \eta^i a_i^j \cdot Z_j \in \mathbb{P}^N.$$

(C) The m-plane $x \in \mathbb{G}_m^\circ$, which is expressed as $(a_i^j)_{i,j}$ under (1), is also given by the set of points $(Z^0 : Z^1 : \cdots : Z^N) \in \mathbb{P}^N$ such that

$$\begin{bmatrix} a_0^{m+1} & \cdots & a_m^{m+1} & 1 & & & \mathbf{0} \\ a_0^{m+2} & \cdots & a_m^{m+2} & & 1 & & \\ \vdots & & \vdots & & & \ddots & \\ a_0^N & \cdots & a_m^N & \mathbf{0} & & & 1 \end{bmatrix} \begin{bmatrix} Z^0 \\ Z^1 \\ \vdots \\ Z^N \end{bmatrix} = \mathbf{0}.$$

Let $V_{\mathbb{G}(m,\mathbb{P}^N)} := \mathbb{P}_*(\mathcal{S}_{\mathbb{G}(m,\mathbb{P}^N)})$ in $\mathbb{G}(m,\mathbb{P}^N) \times (\mathbb{P}^N)^\vee$, which is the universal family of $\mathbb{G}(N-m-1, (\mathbb{P}^N)^\vee)$. Then we have an identification

$$V_{\mathbb{G}_m^\circ} \simeq \mathbb{G}_m^\circ \times \mathbb{P}^{N-m-1} \simeq \mathrm{Hom}(Q^\vee, S^\vee) \times \mathbb{P}^{N-m-1}.$$

Regarding $(\zeta_{m+1} : \cdots : \zeta_N)$ as homogeneous coordinates on $\mathbb{P}^{N-m-1} = \mathbb{P}_*(S)$, we can parametrize $V_{\mathbb{G}_m^\circ} \to (\mathbb{P}^N)^\vee$ by sending $((a_i^j)_{i,j}, (\zeta_{m+1} : \cdots : \zeta_N))$ to the hyperplane defined by the homogeneous polynomial

$$(3) \qquad \sum_{0 \leqslant i \leqslant m, m+1 \leqslant j \leqslant N} \zeta_j a_i^j \cdot Z^i + \sum_{m+1 \leqslant j \leqslant N} \zeta_j \cdot Z^j.$$

2.2. Parametrization of expanding maps.

Let $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$ be a subvariety with $m \geqslant 0$. We will give a local parametrization of the expanding map $\gamma : \mathcal{X} \dashrightarrow \mathbb{G}(m^+, \mathbb{P}^N)$ around a general point $x_o \in \mathcal{X}$ in the following two steps.

Step 1. Changing the homogeneous coordinates $(Z^0 : \cdots : Z^N)$ on $\mathbb{P}^N$, we can assume that $x_o \in \mathbb{G}(m, \mathbb{P}^N)$ and $\gamma(x_o) \in \mathbb{G}(m^+, \mathbb{P}^N)$ are linear subvarieties of dimensions m and m^+ such that

$$(4) \qquad x_o = (Z^{m+1} = \cdots = Z^N = 0),$$

$$(5) \qquad \gamma(x_o) = (Z^{m^++1} = \cdots = Z^N = 0),$$

in $\mathbb{P}^N$. As in §2.1, let us consider the open subset $\mathbb{G}_m^\circ \subset \mathbb{G}(m, \mathbb{P}^N)$, and take a system of regular parameters $z^1, \ldots, z^{\dim(\mathcal{X})}$ of the regular local ring $\mathcal{O}_{\mathcal{X}, x_o}$. Then, under the identification (1), $\mathcal{X} \cap \mathbb{G}_m^\circ$ is parametrized around x_o by

$$\sum_{0 \leqslant i \leqslant m, m+1 \leqslant j \leqslant N} f_i^j \cdot \eta^i \otimes \zeta_j = (f_i^j)_{i,j}$$

with regular functions f_i^j's. From (4), we have $f_i^j(x_o) = 0$. For a general point $x \in \mathcal{X}$ near x_o, we identify Q with $\mathcal{Q}_\mathcal{X} \otimes k(x)$, and S with $\mathcal{S}_\mathcal{X} \otimes k(x)$. Then the linear map $T_x \mathcal{X} \hookrightarrow T_{\gamma(x)} \mathbb{G}(m, \mathbb{P}^N) = \mathrm{Hom}(Q^\vee, S^\vee)$ is represented by

$$(6) \qquad \frac{\partial}{\partial z^e} \mapsto \sum_{0 \leqslant i \leqslant m, m+1 \leqslant j \leqslant N} f_{i,z^e}^j(x) \cdot \eta^i \otimes \zeta_j \qquad (1 \leqslant e \leqslant \dim(\mathcal{X})),$$

where $f_{i,z^e}^j := \partial f_i^j / \partial z^e$. Therefore $\mathrm{Hom}(\mathrm{Hom}(S,Q),Q) \to \mathrm{Hom}(T_x\mathcal{X},Q)$ is represented by

$$\zeta^j \otimes \eta_i \otimes \eta^{i'} \mapsto \sum_{1 \leqslant e \leqslant \dim(\mathcal{X})} f_{i,z^e}^j(x) \cdot dz^e \otimes \eta^{i'} \quad (0 \leqslant i, i' \leqslant m, m+1 \leqslant j \leqslant N).$$

Since $S \to \mathrm{Hom}(\mathrm{Hom}(S,Q),Q)$ is given by $\zeta^j \mapsto \zeta^j \otimes \sum_i (\eta_i \otimes \eta^i)$, it follows that $\varphi_x : S \to \mathrm{Hom}(T_x\mathcal{X},Q)$ is represented by

$$(7) \qquad \varphi_x(\zeta^j) = \sum_{0 \leqslant i \leqslant m, 1 \leqslant e \leqslant \dim(\mathcal{X})} f_{i,z^e}^j(x) \cdot dz^e \otimes \eta^i \qquad (m+1 \leqslant j \leqslant N).$$

Recall that m^+ is the integer such that $N - m^+ = \dim(\ker \varphi_x)$, which implies that $\dim(\varphi_x(S)) = m^+ - m$. By (5), we have $\varphi_{x_o}(\zeta^{m^++1}) = \cdots = \varphi_{x_o}(\zeta^N) = 0$. It follows that $\varphi_x(\zeta^{m+1}), \ldots, \varphi_x(\zeta^{m^+})$ give a basis of the vector space $\varphi_x(S)$, and that

$$(8) \qquad \varphi_x(\zeta^\mu) = - \sum_{m+1 \leqslant \nu \leqslant m^+} g_\nu^\mu(x) \cdot \varphi_x(\zeta^\nu) \qquad (m^+ + 1 \leqslant \mu \leqslant N)$$

with regular functions g_ν^μ's. As a result, we have

$$(9) \qquad f_{i,z^e}^\mu = - \sum_{m+1 \leqslant \nu \leqslant m^+} g_\nu^\mu f_{i,z^e}^\nu \qquad (0 \leqslant i \leqslant m, 1 \leqslant e \leqslant \dim(\mathcal{X}))$$

for $m^+ + 1 \leqslant \mu \leqslant N$. Since $\varphi_{x_o}(\zeta^\mu) = 0$ for $m^+ + 1 \leqslant \mu \leqslant N$, in this setting, it follows that $g_\nu^\mu(x_o) = 0$ and that

$$(10) \qquad f_{i,z^e}^\mu(x_o) = 0 \qquad (0 \leqslant i \leqslant m, 1 \leqslant e \leqslant \dim(\mathcal{X}))$$

for $m^+ + 1 \leqslant \mu \leqslant N$.

Lemma 2.8. *Let $\mathcal{X} \subset \mathbb{G}(m,\mathbb{P}^N)$ be of dimension > 0 with $m < N$. For the integer m^+ given with $\gamma : \mathcal{X} \dashrightarrow \mathbb{G}(m^+, \mathbb{P}^N)$, we have $m^+ > m$.*

Proof. Assume $m^+ = m$. Then, as above, we have $\dim(\varphi_x(S)) = 0$, which means that $\varphi_x(\zeta^j) = 0$ for any $m+1 \leqslant j \leqslant N$. Then we have $f_{i,z^e}^j = 0$ for any i, j, e. This contradicts that f_i^j's are regular functions parametrizing the embedding $\mathcal{X} \hookrightarrow \mathbb{G}(m, \mathbb{P}^N)$ around the point x_o. $\square$

Step 2. We set $\mathcal{Y} \subset \mathbb{G}(m^+, \mathbb{P}^N)$ to be the closure of $\mathcal{X}$ under γ. As in §2.1(C), we set $V_{\mathbb{G}(m,\mathbb{P}^N)} := \mathbb{P}_*(\mathcal{S}_{\mathbb{G}(m,\mathbb{P}^N)})$ and consider the generalized conormal morphism

$$\gamma^* \bar{\pi}|_{\mathcal{X}^\circ} : \mathbb{P}_*(\ker \varphi|_{\mathcal{X}^\circ}) \subset V_{\mathbb{G}(m,\mathbb{P}^N)} \to (\mathbb{P}^N)^\vee$$

given in Definition 2.7. Let ℓ_x be the fiber of $\mathbb{P}_*(\ker \varphi|_{\mathcal{X}^\circ}) \to \mathcal{X}^\circ$ at x, and let $v \in \ell_x$ be a point. Here v is expressed as $((f_i^j(x))_{i,j}, (s_{m+1} : \cdots : s_N))$. Since

$\sum_{m+1\leqslant j\leqslant N} s_j \cdot \varphi_x(\zeta^j) = \varphi_x(\sum_{m+1\leqslant j\leqslant N} s_j\zeta^j) = 0$, the equality (8) implies

$$\sum_{m+1\leqslant \nu\leqslant m^+} s_\nu \cdot \varphi_x(\zeta^\nu) = - \sum_{m^++1\leqslant \mu\leqslant N} s_\mu \cdot \varphi_x(\zeta^\mu)$$

$$= \sum_{m+1\leqslant \nu\leqslant m^+, m^++1\leqslant \mu\leqslant N} s_\mu g_\nu^\mu(x) \cdot \varphi_x(\zeta^\nu).$$

Thus $s_\nu = \sum_{m^++1\leqslant \mu\leqslant N} s_\mu g_\nu^\mu$ for $m+1 \leqslant \nu \leqslant m^+$. Then it follows from (3) that each point $v \in \ell_x$ is sent to the hyperplane $\gamma^*\bar{\pi}(v) \in (\mathbb{P}^N)^\vee$ which is defined by the homogeneous polynomial

$$(11) \quad \sum_{\substack{0\leqslant i\leqslant m \\ m+1\leqslant \nu\leqslant m^+}} \sum_{m^++1\leqslant \mu\leqslant N} s_\mu g_\nu^\mu f_i^\nu(x) \cdot Z^i + \sum_{\substack{0\leqslant i\leqslant m \\ m^++1\leqslant \mu\leqslant N}} s_\mu f_i^\mu(x) \cdot Z^i$$

$$+ \sum_{\substack{m+1\leqslant \nu\leqslant m^+ \\ m^++1\leqslant \mu\leqslant N}} s_\mu g_\nu^\mu(x) \cdot Z^\nu + \sum_{m^++1\leqslant \mu\leqslant N} s_\mu \cdot Z^\mu.$$

Note that, for the m^+-plane $\gamma(x) \subset \mathbb{P}^N$, the image $\gamma^*\bar{\pi}(\ell_x)$ is equal to $\gamma(x)^* \subset (\mathbb{P}^N)^\vee$, the set of hyperplanes containing $\gamma(x)$.

Now, the parametrization of the expanding map $\gamma : \mathcal{X} \dashrightarrow \mathcal{Y}$ is obtained, as follows: Let $\mathcal{Q}_{\mathbb{G}(m^+,\mathbb{P}^N)}$ and $\mathcal{S}_{\mathbb{G}(m^+,\mathbb{P}^N)}$ be the universal quotient bundle and subbundle of rank $m^+ + 1$ and $N - m^+$ on $\mathbb{G}(m^+,\mathbb{P}^N)$. In a similar way to §2.1(A), we take a standard open subset $\mathbb{G}_{m^+}^\circ$ as the set of m^+-planes not intersecting the $(N - m^+ - 1)$-plane $(Z^0 = \cdots = Z^{m^+} = 0)$. Then $\mathcal{Q}_{\mathbb{G}_{m^+}^\circ}$ and $\mathcal{S}_{\mathbb{G}_{m^+}^\circ}^\vee$ are equal to $Q_+ \otimes \mathcal{O}_{\mathbb{G}_{m^+}^\circ}$ and $S_+^\vee \otimes \mathcal{O}_{\mathbb{G}_{m^+}^\circ}$, for vector spaces

$$Q_+ = \bigoplus_{0\leqslant \lambda\leqslant m^+} K \cdot q^\lambda \quad \text{and} \quad S_+^\vee = \bigoplus_{m^++1\leqslant \mu\leqslant N} K \cdot s_\mu,$$

where q^λ and s_μ correspond to Z^λ and Z_μ.

In this setting, by (3) and (11), $\gamma(x) \in \mathbb{G}_{m^+}^\circ = \text{Hom}(Q_+^\vee, S_+^\vee)$ is expressed as

$$(12) \quad \sum_{\substack{0\leqslant i\leqslant m \\ m^++1\leqslant \mu\leqslant N}} (f_i^\mu + \sum_{m+1\leqslant \nu\leqslant m^+} g_\nu^\mu f_i^\nu)(x) \cdot q^i \otimes s_\mu + \sum_{\substack{m+1\leqslant \nu\leqslant m^+ \\ m^++1\leqslant \mu\leqslant N}} g_\nu^\mu(x) \cdot q^\nu \otimes s_\mu$$

for a point $x \in \mathcal{X}$ near x_o.

2.3. Composition of expanding maps and shrinking maps.

We set $\mathcal{X} \subset \mathbb{G}(m,\mathbb{P}^N)$ to be a quasi-projective smooth variety, and set $\mathcal{Y} \subset \mathbb{G}(m^+,\mathbb{P}^N)$ to be the closure of the expanding map $\gamma : \mathcal{X} \dashrightarrow \mathbb{G}(m^+,\mathbb{P}^N)$. For this $\mathcal{Y}$, we will investigate the homomorphism Φ given in Definition 2.3, by considering

the pull-back of Φ via γ. Now we have the following commutative diagram:

$$(13) \qquad \gamma^*\mathcal{Q}_{\mathcal{Y}}^{\vee} \longrightarrow \mathcal{H}om(\mathcal{H}om(\gamma^*\mathcal{Q}_{\mathcal{Y}}^{\vee}, \gamma^*\mathcal{S}_{\mathcal{Y}}^{\vee}), \gamma^*\mathcal{S}_{\mathcal{Y}}^{\vee}) \longrightarrow \mathcal{H}om(\gamma^*T_{\mathcal{Y}}, \gamma^*\mathcal{S}_{\mathcal{Y}}^{\vee})$$

with $\gamma^*\Phi$ across the top, $-\circ d\gamma$ and Ψ mapping down to $\mathcal{H}om(T_{\mathcal{X}}, \gamma^*\mathcal{S}_{\mathcal{Y}}^{\vee})$,

where $\Psi : \gamma^*\mathcal{Q}_{\mathcal{Y}}^{\vee} \xrightarrow{\gamma^*\Phi} \mathcal{H}om(\gamma^*T_{\mathcal{Y}}, \gamma^*\mathcal{S}_{\mathcal{Y}}^{\vee}) \to \mathcal{H}om(T_{\mathcal{X}}, \gamma^*\mathcal{S}_{\mathcal{Y}}^{\vee})$ is the composite homomorphism, and

$$d\gamma : T_{\mathcal{X}} \to \gamma^*T_{\mathbb{G}(m^+, \mathbb{P}^N)} \simeq \mathcal{H}om(\gamma^*\mathcal{Q}_{\mathcal{Y}}^{\vee}, \gamma^*\mathcal{S}_{\mathcal{Y}}^{\vee})$$

is the homomorphism of tangent bundles induced by γ.

We recall that a rational map $f : A \dashrightarrow B$ of varieties is said to be *separable* if the field extension $K(A)/K(f(A))$ is separably generated. Here, the following three conditions are equivalent: (i) f is separable; (ii) the linear map $d_x f : T_x A \to T_{f(x)} f(A)$ of Zariski tangent spaces is surjective for general $x \in A$; (iii) a general fiber of f is reduced. In characteristic zero, every rational map must be separable. A rational map is said to be *inseparable* if it is not separable.

Remark 2.9. If γ is separable, then we have

$$\ker \gamma^*\Phi|_{\mathcal{X}^\circ} = \ker \Psi|_{\mathcal{X}^\circ}$$

for a certain open subset $\mathcal{X}^\circ \subset \mathcal{X}$. This is because the vertical arrow in (13), $\mathrm{Hom}(T_{\gamma(x)}\mathcal{Y}, \mathcal{S}_{\mathcal{Y}}^{\vee} \otimes \gamma(x)) \to \mathcal{H}om(T_x\mathcal{X}, \mathcal{S}_{\mathcal{Y}}^{\vee} \otimes \gamma(x))$, is injective at a general point $x \in \mathcal{X}$.

Let $x_o \in \mathcal{X}$ be a general point. In the setting of §2.2, for a point $x \in \mathcal{X}$ near x_o, it follows from (12) that $d_x\gamma : T_x\mathcal{X} \to T_{\gamma(x)}\mathbb{G}(m^+, \mathbb{P}^N)$ is represented by

$$(14)$$
$$\frac{\partial}{\partial z^e} \mapsto \sum_{\substack{0 \leqslant i \leqslant m \\ m^+ + 1 \leqslant \mu \leqslant N}} \sum_{m+1 \leqslant \nu \leqslant m^+} g_{\nu, z^e}^{\mu} f_i^{\nu}(x) \cdot q^i \otimes s_\mu + \sum_{\substack{m+1 \leqslant \nu \leqslant m^+ \\ m^+ + 1 \leqslant \mu \leqslant N}} g_{\nu, z^e}^{\mu}(x) \cdot q^\nu \otimes s_\mu,$$

where we apply the following equality obtained by (9):

$$(f_i^{\mu} + \sum_{m+1 \leqslant \nu \leqslant m^+} g_\nu^{\mu} f_i^{\nu})_{z^e} = \sum_{m+1 \leqslant \nu \leqslant m^+} g_{\nu, z^e}^{\mu} f_i^{\nu}.$$

Then, from the diagram (13), the linear map $\Psi_x : Q_+^\vee \to \mathrm{Hom}(T_x\mathcal{X}, S_+^\vee)$ is represented by

$$\Psi_x(q_i) = \sum_{\substack{1 \leqslant e \leqslant \dim(\mathcal{X})\ m+1 \leqslant \nu \leqslant m^+ \\ m^++1 \leqslant \mu \leqslant N}} \sum g_{\nu,z^e}^\mu f_i^\nu(x) \cdot dz^e \otimes s_\mu \qquad (0 \leqslant i \leqslant m),$$

(15)

$$\Psi_x(q_\nu) = \sum_{\substack{1 \leqslant e \leqslant \dim(\mathcal{X}) \\ m^++1 \leqslant \mu \leqslant N}} g_{\nu,z^e}^\mu(x) \cdot dz^e \otimes s_\mu \qquad (m+1 \leqslant \nu \leqslant m^+).$$

Lemma 2.10. *The ranks of the above linear maps are obtained as follows.*

(a) $\mathrm{rk}\, d_x\gamma$ *is equal to the rank of the* $\dim(\mathcal{X}) \times (N - m^+) \cdot (m^+ - m)$ *matrix*

$$\begin{bmatrix} g_{m+1,z^1}^{m^++1}(x) & \cdots & g_{\nu,z^1}^\mu(x) & \cdots & g_{m^+,z^1}^N(x) \\ \vdots & & \vdots & & \vdots \\ g_{m+1,z^{\dim(\mathcal{X})}}^{m^++1}(x) & \cdots & g_{\nu,z^{\dim(\mathcal{X})}}^\mu(x) & \cdots & g_{m^+,z^{\dim(\mathcal{X})}}^N(x) \end{bmatrix}.$$

(b) $\mathrm{rk}\, \Psi_x$ *is equal to the rank of the* $(m^+ - m) \times (N - m^+) \cdot \dim(\mathcal{X})$ *matrix*

$$\begin{bmatrix} g_{m+1,z^1}^{m^++1}(x) & \cdots & g_{m+1,z^e}^\mu(x) & \cdots & g_{m+1,z^{\dim(\mathcal{X})}}^N(x) \\ \vdots & & \vdots & & \vdots \\ g_{m^+,z^1}^{m^++1}(x) & \cdots & g_{m^+,z^e}^\mu(x) & \cdots & g_{m^+,z^{\dim(\mathcal{X})}}^N(x) \end{bmatrix}.$$

Proof. Considering the matrix description of (14), we find that, each column vector not belonging to the matrix of (a) is expressed as

$$\begin{bmatrix} \sum_{m+1 \leqslant \nu \leqslant m^+} g_{\nu,z^1}^\mu f_i^\nu(x) \\ \vdots \\ \sum_{m+1 \leqslant \nu \leqslant m^+} g_{\nu,z^{\dim(X)}}^\mu f_i^\nu(x) \end{bmatrix}.$$

This vector is linearly dependent on column vectors of the matrix of (a); hence the assertion of (a) follows. In the same way, considering the matrix description of (15), we have the assertion of (b). $\square$

Proposition 2.11. $Q_{\mathcal{X}^\circ}^\vee \subset \ker \Psi|_{\mathcal{X}^\circ}$ *for a certain open subset* $\mathcal{X}^\circ \subset \mathcal{X}$

Proof. It is sufficient to show that $Q_{\mathcal{X}}^\vee \otimes k(x_o) \subset \ker \Psi_{x_o}$ for a general point $x_o \in \mathcal{X}$. In the setting of §2.2, since $f_i^j(x_o) = 0$, it follows from (15) that

$$\Psi_{x_o}(q_0) = \cdots = \Psi_{x_o}(q_m) = 0.$$

This implies that $Q_{\mathcal{X}}^\vee \otimes k(x_o) \subset \ker \Psi_{x_o}$ in $H^0(\mathbb{P}^N, \mathcal{O}(1))$. $\square$

Let $\sigma = \sigma_{\mathcal{Y}/\mathbb{G}(m^+,\mathbb{P}^N)} : \mathcal{Y} \dashrightarrow \mathbb{G}(m_0,\mathbb{P}^N)$ be the shrinking map of $\mathcal{Y} \subset \mathbb{G}(m^+,\mathbb{P}^N)$, where we set

$$m_0 := (m^+)^-.$$

Corollary 2.12. *Assume that* γ *is separable and assume that* $\mathrm{rk}\, \Psi_x$ *is of rank* $m^+ - m$ *for general* $x \in \mathcal{X}$*. Then* $m_0 = m$ *and* $Q|_{\mathcal{X}^\circ}^\vee = \ker \gamma^*\Phi|_{\mathcal{X}^\circ}$ *for a certain open subset* $\mathcal{X}^\circ$*; hence* $\sigma \circ \gamma|_{\mathcal{X}^\circ}$ *is an identity map of* $\mathcal{X}^\circ \subset \mathcal{X}$*.*

Proof. From Proposition 2.11, we have $\mathcal{Q}|_{\mathcal{X}^\circ}^\vee \subset \ker \Psi|_{\mathcal{X}^\circ}$. Since $(m^+ + 1) - \mathrm{rk}\,\Psi_x = (m^+ + 1) - (m^+ - m) = m + 1$, we have $\mathcal{Q}|_{\mathcal{X}^\circ}^\vee = \ker \Psi|_{\mathcal{X}^\circ}$. It follows from Remark 2.9 that

$$\mathcal{Q}|_{\mathcal{X}^\circ}^\vee = \ker \gamma^* \Phi|_{\mathcal{X}^\circ}.$$

We recall that, by universality, the morphism $\sigma \circ \gamma : \mathcal{X}^\circ \to \mathbb{G}(m_0, \mathbb{P}^N)$ is induced from $\ker \gamma^* \Phi|_{\mathcal{X}^\circ} \subset H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee \otimes \mathcal{O}_{\mathcal{X}^\circ}$. Therefore $\sigma \circ \gamma$ coincides with the original embedding $\mathcal{X} \hookrightarrow \mathbb{G}(m, \mathbb{P}^N)$. $\square$

For the universal family $U_\mathcal{X} \subset \mathcal{X} \times \mathbb{P}^N$, we define a rational map

$$\tilde{\gamma} : U_\mathcal{X} \dashrightarrow \mathcal{Y} \times \mathbb{P}^N,$$

by sending $(x, x') \in U_\mathcal{X}$ with $x \in \mathcal{X}$ and $x' \in \mathbb{P}^N$ to $(\gamma(x), x') \in \mathcal{Y} \times \mathbb{P}^N$. Let $\sigma^* U_{\mathbb{G}(m_0, \mathbb{P}^N)} \subset \mathcal{Y} \times \mathbb{P}^N$ be the closure of the pull-back of $U_{\mathbb{G}(m_0, \mathbb{P}^N)}$ under σ.

Corollary 2.13. *Assume that γ is separable. Then the image of $U_\mathcal{X}$ under $\tilde{\gamma}$ is contained in $\sigma^* U_{\mathbb{G}(m_0, \mathbb{P}^N)}$, and hence we have the following inclusion of linear varieties of $\mathbb{P}^N$:*

$$x \subset \sigma \circ \gamma(x) \subset \gamma(x).$$

Proof. As in Remark 2.9, $\ker \gamma^* \Phi|_{\mathcal{X}^\circ} = \ker \Psi|_{\mathcal{X}^\circ}$. Then Proposition 2.11 implies that $U_{\mathcal{X}^\circ}$ is contained in $\gamma^* \sigma^* U_{\mathbb{G}(m_0, \mathbb{P}^N)} = \mathbb{P}_* \ker(\gamma^* \Phi|_{\mathcal{X}^\circ})$. Hence we have $\tilde{\gamma}(U_{\mathcal{X}^\circ}) \subset \sigma^* U_{\mathbb{G}(m_0, \mathbb{P}^N)}$. $\square$

It is known that, in characteristic two, the Gauss map of every curve is inseparable. This also happens to the expanding map, as follows.

Lemma 2.14. *Assume that $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$ is a curve, and assume that γ is generically finite. If the characteristic is two, then γ is inseparable.*

Proof. Let $x_o \in \mathcal{X}$ be a general point. In the setting of §2.2, since $\mathcal{X}$ is a curve, it is locally parametrized around x_o by one parameter z. It follows from (9) and $g_\nu^\mu(x_o) = 0$ that

$$f_{i,z,z}^\mu(x_o) = - \sum_{m+1 \leqslant \nu \leqslant m^+} g_{\nu,z}^\mu f_{i,z}^\nu(x_o).$$

In characteristic two, we have $f_{i,z,z}^\mu(x_o) = 0$. Since the above formula vanishes for each μ and i, it follows from (7) that $\sum_{m+1 \leqslant \nu \leqslant m^+} g_{\nu,z}^\mu(x_o) \cdot \varphi_{x_o}(\zeta^\nu) = 0$. Since $\varphi_{x_o}(\zeta^\nu)$'s are linearly independent, $g_{\nu,z}^\mu(x_o) = 0$. By Lemma 2.10, $\mathrm{rk}\, d_{x_o} \gamma = 0$, that is to say, γ is inseparable. $\square$

3. Duality with expanding maps and shrinking maps

Let σ be the shrinking map from a subvariety $\mathcal{Y} \subset \mathbb{G}(M, \mathbb{P}^N)$ to $\mathbb{G}(m_0, \mathbb{P}^N)$ with integers M, m_0 ($M \geqslant m_0$), as in Definition 2.3 (we set $m_0 := M^-$). Let $\mathcal{X}_0 \subset \mathbb{G}(m_0, \mathbb{P}^N)$ be the closure of the image of the map σ, and let π_0 be the projection from the universal family $U_{\mathcal{X}_0} \subset \mathcal{X}_0 \times \mathbb{P}^N$ of $\mathcal{X}_0$ to $\mathbb{P}^N$. We set

$$\sigma^* U_{\mathcal{X}_0} \subset \mathcal{Y} \times \mathbb{P}^N$$

to be the closure of the pull-back of $U_{\mathcal{X}_0}$ under σ, and set $\sigma^*\pi_0$ to be the projection from $\sigma^*U_{\mathcal{X}_0}$ to $\mathbb{P}^N$. Note that these constructions of σ and $\sigma^*\pi_0$ depend only on $\mathcal{Y}$. For a subvariety $X \subset \mathbb{P}^N$, we set

$$\Gamma(X) := \overline{\{\, (\mathbb{T}_x X, x) \in \mathbb{G}(M, \mathbb{P}^N) \times \mathbb{P}^N \mid x \in X^{sm} \,\}},$$

the incidence variety of embedded tangent spaces and their contact points, where X^{sm} is the smooth locus.

Theorem 3.1 (Main theorem). *Let N, M be integers with $0 < M < N$, let $X \subset \mathbb{P}^N$ be an M-dimension closed subvariety, and let $\mathcal{Y} \subset \mathbb{G}(M, \mathbb{P}^N)$ be a closed subvariety. We set σ as above, and so on. Then the following are equivalent:*

(a) *The Gauss map $\gamma = \gamma_X : X \dashrightarrow \mathbb{G}(M, \mathbb{P}^N)$ is separable, and the closure of its image is equal to $\mathcal{Y}$.*

(b) *$\Gamma(X) = \sigma^*U_{\mathcal{X}_0}$ in $\mathbb{G}(M, \mathbb{P}^N) \times \mathbb{P}^N$.*

(c) *$\sigma^*\pi_0 : \sigma^*U_{\mathcal{X}_0} \to \mathbb{P}^N$ is separable and generically finite, and its image is equal to X (in particular, the image is of dimension M.)*

(c') *$\sigma^*\pi_0 : \sigma^*U_{\mathcal{X}_0} \to \mathbb{P}^N$ is separable and its image is equal to X, and σ is separable.*

Corollary 3.2. *Assume that one of the conditions (a-c') holds. Then $m_0 = M - \dim(\mathcal{Y})$, and the diagram*

(16)
$$
\begin{array}{ccc}
U_{\mathcal{X}_0} & \xrightarrow{\ \pi_0\ } & X \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma_X} \\
\mathcal{X}_0 & \underset{\gamma_{\mathcal{X}_0}^-}{\dashrightarrow} & \mathcal{Y}
\end{array}
$$

is commutative, where $\gamma_{\mathcal{X}_0}$ is the expanding map of $\mathcal{X}_0$ and is indeed a birational map whose inverse is the shrinking map σ, and where $\sigma(y) \in \mathcal{X}_0$ corresponds to the closure of the fiber $\gamma_X^{-1}(y) \subset \mathbb{P}^N$ for a general point $y \in \mathcal{Y}$ (see also Remarks 4.5, 4.6).

In §3.1 we will show the implication (a) $\Rightarrow$ (b) of Theorem 3.1, which leads to the linearity of general fibers of separable Gauss maps (Corollary 3.6). In §3.2 we will show (c') $\Rightarrow$ (a), and complete the proof of Theorem 3.1. Here the implication (c) or (c') $\Rightarrow$ (a) gives a generalization of the characterization of Gauss images (Corollary 3.14). We note that both implications (a) $\Rightarrow$ (b) and (c') $\Rightarrow$ (a) will be discussed in the same framework, given in §2 (see Remark 3.11).

Remark 3.3. (b) $\Rightarrow$ (a) of Theorem 3.1 holds, as follows: For $\tilde{\gamma} : X \dashrightarrow \Gamma(X) : x \mapsto (\gamma(x), x)$, since $\sigma^*U_{\mathcal{X}_0} \to \mathcal{Y}$ is separable and since $\tilde{\gamma}$ is birational, the composite map γ is separable. In addition, (b) $\Rightarrow$ (c) holds, since $\sigma^*\pi_0$ is identified with the birational projection $\Gamma(X) \to X$ under the assumption. On the other hand, (c) $\Rightarrow$ (c') holds, since, if $\sigma^*\pi_0 : \sigma^*U_{\mathcal{X}_0} \to \mathbb{P}^N$ is separable and generically finite onto its image, then $\sigma^*U_{\mathcal{X}_0} \dashrightarrow U_{\mathcal{X}_0}$ is separable, and then so is σ.

3.1. **Separable Gauss maps and shrinking maps.** In this subsection, we consider the Gauss map $X \dashrightarrow \mathbb{G}(\dim(X), \mathbb{P}^N)$ of a quasi-projective smooth subvariety $X \subset \mathbb{P}^N$, which coincides with the expanding map of X, as in Remark 2.2. We denote by γ the map, and by $\mathcal{Y} \subset \mathbb{G}(\dim(X), \mathbb{P}^N)$ the closure of the image of γ. In the setting of §2.3 with $m = 0$, we have a natural homomorphism $\xi : \gamma^* \mathcal{Q}_{\mathcal{Y}}^\vee \to T_X$ with the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \gamma^* \mathcal{Q}_{\mathcal{Y}}^\vee & \longrightarrow & H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee \otimes \mathcal{O}_X & \longrightarrow & \gamma^* \mathcal{S}_{\mathcal{Y}}^\vee & \longrightarrow & 0 \\
& & \xi \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & T_X(-1) & \longrightarrow & T_{\mathbb{P}^N}(-1)|_X & \longrightarrow & N_{X/\mathbb{P}^N}(-1) & \longrightarrow & 0.
\end{array}
$$

Recall that Ψ is a homomorphism given in the diagram (13) in §2.3.

Proposition 3.4. *Assume that $m = 0$. Then we have an equality,*

$$
\ker \Psi|_{X^\circ} = \ker d\gamma(-1) \circ \xi|_{X^\circ}
$$

for a certain open subset $X^\circ \subset X$.

Proof. It is sufficient to prove

$$
\ker \Psi \otimes k(x_o) = \ker(d\gamma(-1) \circ \xi) \otimes k(x_o)
$$

for a general point $x_o \in X$. In the setting of §2.2, changing coordinates on $\mathbb{P}^N$, we can assume $x_o = (1 : 0 : \cdots : 0)$ and $\gamma(x_o) = (Z^{\dim(X)+1} = \cdots = Z^N = 0)$. Then, by taking $z^1, \ldots, z^{\dim(X)}$ as $f_0^1, \ldots, f_0^{\dim(X)}$, the original embedding $X \hookrightarrow \mathbb{P}^N$ can be locally parametrized by $(1 : z^1 : \cdots : z^{\dim(X)} : f_0^{\dim(X)+1} : \cdots : f_0^N)$. We find

$$
\varphi_x(\zeta^1) = dz^1 \otimes \eta^0, \ \varphi_x(\zeta^2) = dz^2 \otimes \eta^0, \ \cdots, \ \varphi_x(\zeta^{\dim(X)}) = dz^{\dim(X)} \otimes \eta^0
$$

in (7). Hence we have $g_\nu^\mu = -f_{0,z^\nu}^\mu$ in (8). As in (14), the linear map $d_{x_o}\gamma : T_{x_o}X \to T_{\gamma(x_o)}\mathbb{G}(\dim(X), \mathbb{P}^N)$ is represented by

$$
\frac{\partial}{\partial z^e} \mapsto \sum_{\substack{1 \leqslant \nu \leqslant \dim(X) \\ \dim(X)+1 \leqslant \mu \leqslant N}} -f_{0,z^\nu,z^e}^\mu(x_o) \cdot s_\mu \otimes q^\nu \qquad (1 \leqslant e \leqslant \dim(\mathcal{X})).
$$

It follows from (15) that $\Psi_{x_o} : Q_+^\vee \to \operatorname{Hom}(T_x X, S_+^\vee)$ is represented by

$$
\Psi_{x_o}(q_0) = 0, \ \Psi_{x_o}(q_\nu) = \sum_{\substack{1 \leqslant e \leqslant \dim(X) \\ \dim(X)+1 \leqslant \mu \leqslant N}} -f_{0,z^\nu,z^e}^\mu(x_o) \cdot dz^e \otimes s_\mu \quad (1 \leqslant \nu \leqslant \dim(X)).
$$

Since $\xi_{x_o} : Q_+^\vee \to T_{x_o}X$ is obtained by $\xi_{x_o}(q_0) = 0$ and $\xi_{x_o}(q_\nu) = \partial/\partial z^\nu$ with $1 \leqslant \nu \leqslant \dim(X)$, the linear maps $d_{x_o}\gamma \circ \xi_{x_o}$ and Ψ_{x_o} can be identified; in particular, these kernels coincide. $\square$

Theorem 3.5. *The implication* (a) $\Rightarrow$ (b) *of Theorem 3.1 holds.*

Proof. Let $X \subset \mathbb{P}^N$ be a projective variety, let $\mathcal{Y} \subset \mathbb{G}(\dim(X), \mathbb{P}^N)$ be the closure of the image of X under γ, and let $\mathcal{Y} \dashrightarrow \mathbb{G}(m_0, \mathbb{P}^N)$ be the shrinking map with $m_0 := \dim(X)^-$. We apply the previous argument to the quasi-projective variety X^{sm}.

Assume that γ is separable. Then $\ker \gamma^* \Phi|_{X^\circ} = \ker \Psi|_{X^\circ}$ as in Remark 2.9. Therefore Proposition 3.4 implies

$$\ker \gamma^* \Phi|_{X^\circ} = \ker(d\gamma(-1) \circ \xi)|_{X^\circ},$$

where the right hand side is of rank $\dim(X) - \dim \mathcal{Y} + 1$ because of the separability of γ. This implies $m_0 = \dim(X) - \dim(\mathcal{Y})$.

On the other hand, it follows from Corollary 2.13 that $\Gamma(X) \subset \sigma^* U_{\mathcal{X}_0}$. In fact, $\Gamma(X)$ and $\sigma^* U_{\mathcal{X}_0}$ coincide, since both have the same dimension, $\dim(X)$. $\square$

Now, we give the proof of Theorem 1.1; more precisely, we have:

Corollary 3.6. *Assume that the Gauss map $\gamma : X \dashrightarrow \mathcal{Y}$ is separable. Then a general fiber $\gamma^{-1}(y) \subset X^{sm}$ with $y \in \mathcal{Y}$ is equal to the intersection of X^{sm} and the m_0-plane $\sigma(y) \subset \mathbb{P}^N$.*

This implies that the Gauss map is separable if and only if its general fiber is scheme-theoretically (an open subscheme of) a linear subvariety of $\mathbb{P}^N$. The latter condition immediately implies the former one, since a fiber is reduced if it is scheme-theoretically a linear subvariety.

Proof of Corollary 3.6. Recall that, for a general fiber F_y of $\sigma^* U_{\mathcal{X}_0} \to \mathcal{Y}$ at y, the image $\sigma^* \pi_0(F_y)$ is equal to the m_0-plane $\sigma(y) \subset \mathbb{P}^N$. If γ is separable, then Theorem 3.5 implies $\Gamma(X) = \sigma^* U_{\mathcal{X}_0}$, and then the following commutative diagram holds:

$$\begin{array}{ccc} \Gamma(X) & \!\!=\!\!\!=\!\!\!=\!\! & \sigma^* U_{\mathcal{X}_0} \\ \downarrow {\scriptstyle \sigma^* \pi_0} \!\!\swarrow & & \downarrow \\ X & \dashrightarrow_{\gamma} & \mathcal{Y}. \end{array}$$

Thus, $\sigma^* \pi_0$ is birational; moreover, for a general point $y \in \mathcal{Y}$, we have $\gamma^{-1}(y) = X^{sm} \cap \sigma(y)$. $\square$

Combining [26, I, 2.8. Corollary] and Corollary 3.6, we have:

Corollary 3.7. *If the projective variety $X \subset \mathbb{P}^N$ is smooth (and is non-linear), then the separable Gauss map γ is in fact birational onto its image.*

Remark 3.8. In positive characteristic, the Gauss map γ can be *inseparable* (Wallace [25, §7]), and then, in contrast to the characteristic zero case, a general fiber F of γ can be *non-linear*. Several authors gave examples where F of an inseparable γ is not a linear subvariety (Kaji [12, Example 4.1] [13], Rathmann [23, Example 2.13], Noma [21], Fukasawa [2] [3]).

Remark 3.9. If $\dim X = 1$ and the Gauss map γ is separable, then γ is birational. This fact was classically known for plane curves in terms of *dual curves* (for example, see [18, p. 310], [10, §9.4]), and was shown for any curve by Kaji [13, Corollary 2.2].

3.2. **Generalized conormal morphisms.** We denote by γ the expanding map from a closed subvariety $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$ to $\mathbb{G}(m^+, \mathbb{P}^N)$, and by $\mathcal{Y} \subset \mathbb{G}(m^+, \mathbb{P}^N)$ the closure of the image of $\mathcal{X}$ under γ. We consider the generalized conormal morphism $\gamma^* \bar{\pi} : \gamma^* V_{\mathcal{Y}} \to (\mathbb{P}^N)^\vee$ given in Definition 2.7. Let Y be the image of $\gamma^* V_{\mathcal{Y}}$ under $\gamma^* \bar{\pi}$, which is a subvariety of $\mathbb{G}(N-1, \mathbb{P}^N) = (\mathbb{P}^N)^\vee$.

The shrinking map $\sigma_Y = \sigma_{Y/\mathbb{G}(N-1, \mathbb{P}^N)}$ from Y to $\mathbb{G}(N-1-\dim Y, \mathbb{P}^N)$ is identified with the Gauss map $Y \dashrightarrow \mathbb{G}(\dim Y, (\mathbb{P}^N)^\vee)$ which sends $y \in Y$ to $\mathbb{T}_y Y \subset (\mathbb{P}^N)^\vee$. Denoting by $A^* \subset (\mathbb{P}^N)^\vee$ the set of hyperplanes containing a linear subvariety $A \subset \mathbb{P}^N$, we have $\sigma_Y(y)^* = \mathbb{T}_y Y$.

Theorem 3.10. *Let $\mathcal{X}$ be as above. Assume that $\gamma^* \bar{\pi}$ is separable and its image Y is of dimension $N - m - 1$, and assume that γ is separable. Then the shrinking map σ_Y of Y is separable and the closure of its image is equal to $\mathcal{X}$.*

Remark 3.11. By considering the dual of the above statement, it follows that Theorem 3.10 is equivalent to "(c') $\Rightarrow$ (a)" of Theorem 3.1. This is because, in the setting of Theorem 3.1, we have $\sigma_{\mathcal{Y}/\mathbb{G}(M,\mathbb{P}^N)} = \gamma_{\bar{\mathcal{Y}}/\mathbb{G}(N-M-1,(\mathbb{P}^N)^\vee)}$, $\gamma_{X/\mathbb{P}^N} = \sigma_{\bar{X}/\mathbb{G}(N-1,(\mathbb{P}^N)^\vee)}$, and so on (see Remark 2.4).

The following result is essential for the proof of Theorem 3.10, and is indeed a generalization of the Monge-Segre-Wallace criterion. Here we recall that $\Psi_x : Q_+^\vee \to \mathrm{Hom}(T_x \mathcal{X}, S_+^\vee)$ is the linear map given in (13), (15) in §2.3.

Proposition 3.12. *Let $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$, let $v \in \gamma^* V_{\mathcal{Y}}$ be a general point, and let $x \in \mathcal{X}$ be the image of v under $\gamma^* V_{\mathcal{Y}} \to \mathcal{X}$. Then the following holds:*

   (a) $\mathrm{rk}\, d_v \gamma^* \bar{\pi} - (N - m^+ - 1) \leqslant \mathrm{rk}\, d_x \gamma$ *and* $\mathrm{rk}\, d_v \gamma^* \bar{\pi} - (N - m^+ - 1) \leqslant \mathrm{rk}\, \Psi_x$.
   (b) *If* $\mathrm{rk}\, d_v \gamma^* \bar{\pi} = N - m^+ - 1$, *then* $\mathrm{rk}\, d_x \gamma = 0$.
   (c) *If* $\gamma^* \bar{\pi}$ *is separable, then* $\mathbb{T}_{\gamma^* \bar{\pi}(v)} Y \subset x^*$ *in* $(\mathbb{P}^N)^\vee$.

To prove Proposition 3.12, we will describe the linear map $d_{v_o} \gamma^* \bar{\pi}$ for a general point $v_o \in \gamma^* V_{\mathcal{Y}}$, as follows. Under the setting of §2.2, the morphism $\gamma^* \bar{\pi}$ is expressed as (11), where we have $V_{\mathbb{G}_{m^+}^\circ} \simeq \mathbb{G}_{m^+}^\circ \times \mathbb{P}^{N-m^+-1}$ as in §2.1(C). Let $v_o = (x_o, s_o) \in \gamma^* V_{\mathcal{Y}}$ with $x_o \in \mathcal{X}$ and $s_o \in \mathbb{P}^{N-m^+-1}$. Changing homogeneous coordinates on $\mathbb{P}^N$, we can assume that

$$x_o = (Z^{m+1} = \cdots = Z^N = 0) \subset v_o' := \gamma^* \bar{\pi}(v_o) = (Z^N = 0) \text{ in } \mathbb{P}^N.$$

We regard $(s_{m^++1} : \cdots : s_N)$ and $(Z_0 : \cdots : Z_N)$ as homogeneous coordinates on $\mathbb{P}^{N-m^+-1} = \mathbb{P}_*(S_+)$ and $(\mathbb{P}^N)^\vee$. Then, since $v_o' = (Z^N = 0)$, we have

$$s_o = (s_{m^++1} = \cdots = s_{N-1} = 0) \in \mathbb{P}^{N-m^+-1}.$$

For affine coordinates $\bar{s}_\mu := s_\mu / s_N$ on $\{s_N \neq 0\} \subset \mathbb{P}^{N-m^+-1}$, we regard $z^1, \ldots, z^{\dim(\mathcal{X})}, \bar{s}_{m^++1}, \ldots, \bar{s}_{N-1}$ as a system of regular parameters of $\mathcal{O}_{\gamma^* V_{\mathcal{Y}}, v_o}$.

In addition, we set $\bar{s}_N := 1$, and take affine coordinates $\bar{Z}_\alpha := Z_\alpha/Z_N$ on $\{Z_N \neq 0\} \subset (\mathbb{P}^N)^\vee$. Now, for a general point $v = (x, s) \in \gamma^* V_{\mathcal{Y}}$ near v_o, which is expressed as $((f_i^j)_{i,j}, (\bar{s}_{m^++1}, \cdots, \bar{s}_{N-1}))$, it follows from (11) that the linear map $d_v \gamma^* \bar{\pi} : T_v \gamma^* V_{\mathcal{Y}} \to T_{v'}(\mathbb{P}^N)^\vee$ is represented by

(17)

$$
\frac{\partial}{\partial z^e} \mapsto \sum_{\substack{0 \leqslant i \leqslant m \\ m+1 \leqslant \nu \leqslant m^+ \\ m^++1 \leqslant \mu \leqslant N}} \bar{s}_\mu g^\mu_{\nu, z^e} f_i^\nu \cdot \frac{\partial}{\partial \bar{Z}_i} + \sum_{m+1 \leqslant \nu \leqslant m^+} \sum_{m^++1 \leqslant \mu \leqslant N} \bar{s}_\mu g^\mu_{\nu, z^e} \cdot \frac{\partial}{\partial \bar{Z}_\nu},
$$

$$
\frac{\partial}{\partial \bar{s}_{\bar\mu}} \mapsto \sum_{0 \leqslant i \leqslant m} (f_i^{\bar\mu} + \sum_{m+1 \leqslant \nu \leqslant m^+} g^{\bar\mu}_\nu f_i^\nu) \cdot \frac{\partial}{\partial \bar{Z}_i} + \sum_{m+1 \leqslant \nu \leqslant m^+} g^{\bar\mu}_\nu \cdot \frac{\partial}{\partial \bar{Z}_\nu} + \frac{\partial}{\partial \bar{Z}_{\bar\mu}},
$$

for $1 \leqslant e \leqslant \dim(\mathcal{X})$ and $m^+ + 1 \leqslant \bar\mu \leqslant N - 1$. Here the $N - m^+ - 1$ elements $d_v \gamma^* \bar{\pi}(\partial/\partial \bar{s}_{m^++1}), \ldots, d_v \gamma^* \bar{\pi}(\partial/\partial \bar{s}_{N-1})$ are linearly independent, since each of them has $\partial/\partial \bar{Z}_{\bar\mu}$ as its tail term. Moreover, setting a $\dim(\mathcal{X}) \times (m^+ - m)$ matrix

$$
G_{d_v \gamma^* \bar{\pi}} := \begin{bmatrix} \sum_\mu \bar{s}_\mu g^\mu_{m+1, z^1}(x) & \cdots & \sum_\mu \bar{s}_\mu g^\mu_{m^+, z^1}(x) \\ \vdots & & \vdots \\ \sum_\mu \bar{s}_\mu g^\mu_{m+1, z^{\dim(\mathcal{X})}}(x) & \cdots & \sum_\mu \bar{s}_\mu g^\mu_{m^+, z^{\dim(\mathcal{X})}}(x) \end{bmatrix},
$$

we have

$$
\mathrm{rk}\, d_v \gamma^* \bar{\pi} = N - m^+ - 1 + \mathrm{rk}\, G_{d_v \gamma^* \bar{\pi}}.
$$

Proof of Proposition 3.12. (a) From Lemma 2.10(a), it follows that $\mathrm{rk}\, G_{d_{v_o} \gamma^* \bar{\pi}} \leqslant \mathrm{rk}\, d_{x_o} \gamma$. In addition, from Lemma 2.10(b), considering the transpose of the matrix, we have $\mathrm{rk}\, G_{d_{v_o} \gamma^* \bar{\pi}} \leqslant \mathrm{rk}\, \Psi_{x_o}$.

(b) Assume that $\mathrm{rk}\, d_{x_o} \gamma > 0$. Then it follows from Lemma 2.10(a) that $g^\mu_{\nu, z^e}(x_o) \neq 0$ for some μ, ν, e. Hence $G_{d_v \gamma^* \bar{\pi}} \neq 0$ for some v with $v \mapsto x_o$ under $\gamma^* V_{\mathcal{Y}} \to \mathcal{X}$. This implies that $\mathrm{rk}\, d_v \gamma^* \bar{\pi} > N - m^+ - 1$.

(c) Since $f_i^j(x_o) = 0$, the description (17) implies that $\mathrm{im}(d_{v_o} \gamma^* \bar{\pi})$ is contained in the vector subspace of $T_{v'_o}(\mathbb{P}^N)^\vee$ spanned by $\partial/\partial \bar{Z}_{m+1}, \cdots, \partial/\partial \bar{Z}_{N-1}$. If $\gamma^* \bar{\pi}$ is separable, then $\mathbb{T}_{v'_o} Y \subset (Z_0 = \cdots = Z_m = 0)$ in $(\mathbb{P}^N)^\vee$, where the right hand side is equal to x_o^*. $\square$

Recall that $\sigma : \mathcal{Y} \dashrightarrow \mathbb{G}(m_0, \mathbb{P}^N)$ is the shrinking map with $m_0 := (m^+)^-$.

Corollary 3.13. *Assume that $\gamma^* \bar{\pi}$ is separable and its image is of dimension $N - m - 1$, and assume that γ is separable. Then $m_0 = m$ and $\sigma \circ \gamma|_{\mathcal{X}^\circ}$ is an identity map of a certain open subset $\mathcal{X}^\circ \subset \mathcal{X}$.*

Proof. Since $\mathrm{rk}\, d_v \gamma^* \bar{\pi} = N - m - 1$, it follows from Proposition 3.12(a) that $m^+ - m \leqslant \mathrm{rk}\, \Psi_x$ (the equality indeed holds, due to Lemma 2.10(b)). Then Corollary 2.12 implies the result. $\square$

Proof of Theorem 3.10. Corollary 3.13 implies that $m_0 = m$ and that $\sigma \circ \gamma|_{\mathcal{X}^\circ}$ is an identity map of $\mathcal{X}^\circ$. On the other hand, since Y is of dimension $N - m - 1$, in the statement of Proposition 3.12(c), we have $\mathbb{T}_{\gamma^* \bar{\pi}(v)} Y = x^*$ in $(\mathbb{P}^N)^\vee$. Since

σ_Y is identified with the Gauss map $Y \dashrightarrow \mathbb{G}(\dim Y, (\mathbb{P}^N)^\vee)$, it follows that $\sigma_Y(\gamma^*\bar{\pi}(v)) = x$ in $\mathbb{P}^N$. Now the diagram

$$
\begin{array}{ccc}
\gamma^*V_{\mathcal{Y}} \overset{\gamma^*\bar{\pi}}{\dashrightarrow} V_{\mathcal{Y}} \overset{\bar{\pi}}{\longrightarrow} Y & & \\
\downarrow \qquad \downarrow \qquad \vert\ \sigma_Y & & \\
\mathcal{X} \overset{\gamma}{\dashrightarrow} \mathcal{Y} \overset{\sigma}{\dashrightarrow} \mathcal{X} & &
\end{array}
$$

$$\underbrace{\qquad\qquad}_{\text{id}}$$

is commutative. In particular σ_Y is separable. $\qquad\qquad\square$

Proof of Theorem 3.1. (b) $\Rightarrow$ (c) $\Rightarrow$ (c') follows from Remark 3.3. (a) $\Rightarrow$ (b) follows from Theorem 3.5. (c') $\Rightarrow$ (a) follows from Theorem 3.10 and Remark 3.11. $\qquad\qquad\square$

Proof of Corollary 3.2. $\gamma_{\mathcal{X}_0} \circ \sigma$ is identity map due to the dual statement of Corollary 3.13. As in Corollary 3.6, for general y, $\sigma(y)$ corresponds to the closure of the fiber $\gamma_X^{-1}(y)$. Thus the diagram (16) is commutative. $\qquad\square$

From the equivalence (c) $\Leftrightarrow$ (a), we have:

Corollary 3.14. *Let σ be the shrinking map from a closed subvariety $\mathcal{Y} \subset \mathbb{G}(M, \mathbb{P}^N)$ to $\mathbb{G}(M^-, \mathbb{P}^N)$. Then $\mathcal{Y}$ is the closure of a image of a separable Gauss map if and only if $M^- = M - \dim \mathcal{Y}$ holds and $\sigma^*U_{\mathbb{G}(M^-,\mathbb{P}^N)} \to \mathbb{P}^N$ is separable and generically finite onto its image.*

Remark 3.15. In the case where $m = 0$, Proposition 3.12(c) gives the statement of the Monge-Segre-Wallace criterion [8, (2.4)], [19, I-1(4)].

Remark 3.16. For $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$, in the diagram of Definition 2.7, $\gamma^*\bar{\pi}$ can be inseparable even if γ is separable. The reason is as follows: If $m = 0$, then $\gamma^*\bar{\pi}$ coincides with the conormal map $C(X) \to (\mathbb{P}^N)^\vee$ in the original sense. Then, as we mentioned in §1, Kaji [15] and Fukasawa [4] [5] gave examples of non-reflexive varieties (i.e., $\gamma^*\bar{\pi}$'s are inseparable) whose Gauss maps are birational. This implies the assertion.

Considering the dual of the above statement, in the setting for Theorem 3.1, we find that $\sigma^*\pi_0$ can be inseparable even if σ is separable; in other words, separability of σ is not sufficient to give an equivalent condition for separability of the Gauss map of X.

4. DEVELOPABLE PARAMETER SPACES

In this section, we set $\pi = \pi_{\mathcal{X}}$ to be the projection

$$\pi : U_{\mathcal{X}} := U_{\mathbb{G}(m,\mathbb{P}^N)}|_{\mathcal{X}} \to \mathbb{P}^N$$

for a closed subvariety $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$, and set $X := \pi(U_{\mathcal{X}})$ in $\mathbb{P}^N$.

We say that $(\mathcal{X}, X)$ or $\mathcal{X}$ is *developable* if $X = \pi(U_{\mathcal{X}})$ and, for general $x \in \mathcal{X}$, the embedded tangent space $\mathbb{T}_{x'}X$ is the same for any smooth points $x' \in X$

lying in the m-plane $x \subset \mathbb{P}^N$, i.e., the Gauss map γ_X of X is constant on $x \cap X^{sm}$ ([1, 2.2.4]). The variety X is said to be *uniruled (resp. ruled)* by m-planes if π is generically finite (resp. generically bijective).

Note that, in the case where γ_X is separable, there exists a developable parameter space $(\mathcal{X}, X)$ of m-planes with $m > 0$ if and only if the dimension of the image of γ_X is less than $\dim X$ (Remark 4.5).

4.1. **Expanding maps and developable parameter spaces.**

Proposition 4.1. *Let* $\gamma = \gamma_{\mathcal{X}} : \mathcal{X} \dashrightarrow \mathbb{G}(m^+, \mathbb{P}^N)$ *be the expanding map of* $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$. *Then the following holds:*

   (a) $\operatorname{rk} d_u\pi \leqslant m + \dim(\mathcal{X})$ *and* $\operatorname{rk} d_u\pi \leqslant m^+$ *for general* $u \in U_{\mathcal{X}}$.
   (b) *If* $\operatorname{rk} d_u\pi = m$ *for general* $u \in U_{\mathcal{X}}$, *then* $\mathcal{X}$ *is a point.*
   (c) *Assume that* π *is separable, and let* $x \in \mathcal{X}$ *be a general point. Then the* m^+*-plane* $\gamma(x) \subset \mathbb{P}^N$ *is spanned by* $\dim(X)$*-planes* $\gamma_X(u')$ *with smooth points* $u' \in X$ *lying in the* m*-plane* x.

To show Proposition 4.1, we will first describe the linear map $d_u\pi$, as follows. As in §2.1(B), we have $U_{\mathbb{G}_m^\circ} \simeq \mathbb{G}_m^\circ \times \mathbb{P}^m$. Let $u_o = (x_o, \eta_o) \in U_{\mathcal{X}}$ be a general point with $x_o \in \mathcal{X} \cap \mathbb{G}_m^\circ$ and $\eta_o \in \mathbb{P}^m$. Changing homogeneous coordinates on $\mathbb{P}^N$, we can assume that $x_o = (Z^{m+1} = \cdots = Z^N = 0) \subset \mathbb{P}^N$ and $u_o' := \pi(u_o) = (1 : 0 : \cdots : 0) \in \mathbb{P}^N$. Then we have

$$\eta_o = (\bar{\eta}^1 = \cdots = \bar{\eta}^m = 0) \in \mathbb{P}^m.$$

We can also assume $\gamma(x_o) = (Z^{m^+ + 1} = \cdots = Z^N = 0)$. From the expression (2), the projection $\pi : U_{\mathcal{X}} \to \mathbb{P}^N$ sends an element $u = (x, \eta) \in U_{\mathcal{X}}$ near u_o, which is described as $((f_i^j)_{i,j}, (\eta^0 : \cdots : \eta^m))$, to the point

$$\sum_{0 \leqslant i \leqslant m} \eta^i \cdot Z_i + \sum_{0 \leqslant i \leqslant m, m+1 \leqslant j \leqslant N} \eta^i f_i^j \cdot Z_j \in \mathbb{P}^N.$$

Let us take affine coordinates $\bar{\eta}^i := \eta^i / \eta^0$ on $\{\eta^0 \neq 0\} \subset \mathbb{P}^m$, and $\bar{Z}^\alpha := Z^\alpha / Z^0$ on $\{Z^0 \neq 0\} \subset \mathbb{P}^N$. Then we regard $z^1, \ldots, z^{\dim(\mathcal{X})}, \bar{\eta}^1, \ldots, \bar{\eta}^m$ as a system of regular parameters of $\mathcal{O}_{U_{\mathcal{X}}, u_o}$. We set $\bar{\eta}^0 := 1$.

Now, for general $u = (x, \eta) \in U_{\mathcal{X}}$ near u_o, the linear map $d_u\pi : T_u U_{\mathcal{X}} \to T_{u'}\mathbb{P}^N$ is represented by

$$(18) \quad \begin{aligned} \frac{\partial}{\partial \bar{\eta}^{\bar{\imath}}} &\mapsto \frac{\partial}{\partial \bar{Z}^{\bar{\imath}}} + \sum_{m+1 \leqslant j \leqslant N} f_{\bar{\imath}}^j \cdot \frac{\partial}{\partial \bar{Z}^j} && (1 \leqslant \bar{\imath} \leqslant m), \\ \frac{\partial}{\partial z^e} &\mapsto \sum_{0 \leqslant i \leqslant m, m+1 \leqslant j \leqslant N} \bar{\eta}^i f_{i,z^e}^j \cdot \frac{\partial}{\partial \bar{Z}^j} && (1 \leqslant e \leqslant \dim(\mathcal{X})). \end{aligned}$$

Here the m elements $d_u\pi(\partial/\partial\bar{\eta}^1), \ldots, d_u\pi(\partial/\partial\bar{\eta}^m)$ are linearly independent. For a point $u \in U_{\mathcal{X}}$ near u_o such that $u \mapsto x$ under $U_{\mathcal{X}} \to \mathcal{X}$, setting a

$\dim(\mathcal{X}) \times (N - m)$ matrix

$$F_{d_u\pi} := \begin{bmatrix} \sum_i \bar{\eta}^i f_{i,z^1}^{m+1}(x) & \cdots & \sum_i \bar{\eta}^i f_{i,z^1}^N(x) \\ \vdots & & \vdots \\ \sum_i \bar{\eta}^i f_{i,z^{\dim(\mathcal{X})}}^{m+1}(x) & \cdots & \sum_i \bar{\eta}^i f_{i,z^{\dim(\mathcal{X})}}^N(x) \end{bmatrix},$$

we have

$$\operatorname{rk} d_u\pi = \operatorname{rk} F_{d_u\pi} + m.$$

Proof of Proposition 4.1. (a) From (6) in §2.2, we have $\operatorname{rk} F_{d_{u_o}\pi} \leqslant \dim(\mathcal{X})$. From (7), we have $\operatorname{rk} F_{d_{u_o}\pi} \leqslant \operatorname{rk} \varphi_{x_o} = m^+ - m$. Thus the assertion follows.

(b) If $\dim \mathcal{X} > 0$, then $f_{i,z^e}^j(x_o) \neq 0$ for some i, j, e. It follows that $F_{d_u\pi} \neq 0$ for some u with $u \mapsto x_o$. This implies that $\operatorname{rk} d_u\pi > m$.

(c) Changing coordinates, we have that $\gamma(x_o) = (Z^{m^++1} = \cdots = Z^N = 0)$ in $\mathbb{P}^N$. Then the equality (10) implies that, for each $u \in U_{\mathcal{X}}$ near u_o with $u \mapsto x_o$, we have

$$F_{d_u\pi} = \left[\begin{pmatrix} \sum_i \bar{\eta}^i f_{i,z^1}^{m+1}(x_o) & \cdots & \sum_i \bar{\eta}^i f_{i,z^1}^{m^+}(x_o) \\ \vdots & & \vdots \\ \sum_i \bar{\eta}^i f_{i,z^{\dim(\mathcal{X})}}^{m+1}(x_o) & \cdots & \sum_i \bar{\eta}^i f_{i,z^{\dim(\mathcal{X})}}^{m^+}(x_o) \end{pmatrix} \quad \mathbf{0} \right].$$

In addition, we recall that $f_i^j(x_o) = 0$.

Now, we find an inclusion of linear varieties $\gamma_X(\pi(u)) \subset \gamma(x_o)$ in $\mathbb{P}^N$, as follows: Considering the description (18), we have that $\operatorname{im}(d_u\pi)$ is contained in the vector subspace of $T_{\pi(u)}\mathbb{P}^N$ spanned by $\partial/\partial \bar{Z}^1, \cdots, \partial/\partial \bar{Z}^{m^+}$. Since π is separable, $\gamma_X(\pi(u))$ is contained in $\gamma(x_o) = (Z^{m^++1} = \cdots = Z^N = 0)$.

Suppose that there exists an $(m^+ - 1)$-plane $L \subset \mathbb{P}^N$ contained in the m^+-plane $\gamma(x_o)$, such that $\gamma_X(u') \subset L$ holds for each smooth point $u' \in X$ lying in the m-plane x_o. Then we find a contradiction, as follows: Changing coordinates, we can assume

$$L = (Z^{m^+} = 0) \cap \gamma(x_o).$$

Since π is separable and since $\gamma_X(\pi(u)) \subset L$ for each $u \in U_{\mathcal{X}}$ with $u \mapsto x_o$, considering the above matrix $F_{d_u\pi}$ and the description (18) of $d_u\pi$, we have $f_{i,z^e}^{m^+}(x_o) = 0$ for each i, e. Then $\varphi_x(\xi^{m^+}) = 0$ due to (7). This contradicts that a basis of the vector space $\varphi_x(S)$ consists of $\varphi_x(\xi^{m+1}), \cdots, \varphi_x(\xi^{m^+})$. $\qquad\square$

Corollary 4.2. *In the setting of Proposition 3.12, if the maps γ and $\bar{\pi}$ are separable, then we have $\mathbb{T}_{\gamma^*\bar{\pi}(v)}Y \subset (\sigma \circ \gamma(x))^* \subset x^*$ in $(\mathbb{P}^N)^\vee$.*

Proof. From Corollary 2.13, we have $x \subset \sigma \circ \gamma(x)$ in $\mathbb{P}^N$. By applying the dual statement of Proposition 4.1(c) to $\bar{\pi}$ and $\gamma(x)$, the inclusion $\mathbb{T}_{\gamma^*\bar{\pi}(v)}Y \subset (\sigma \circ \gamma(x))^*$ holds. $\qquad\square$

We have the following criterion for developability (cf. [1, 2.2.4]), where recall that m^+ is an integer given with the expanding map $\gamma_{\mathcal{X}} : \mathcal{X} \dashrightarrow \mathbb{G}(m^+, \mathbb{P}^N)$.

Corollary 4.3. *Assume that π is separable. Then* $\dim(X) = m^+$ *if and only if* $(\mathcal{X}, X)$ *is developable. In this case, the following commutative diagram holds:*

(19)
$$
\begin{array}{ccc}
U_{\mathcal{X}} & \xrightarrow{\ \pi\ } & X \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma_X} \\
\mathcal{X} & \dashrightarrow{\scriptstyle \gamma_{\mathcal{X}}} & \mathbb{G}(m^+, \mathbb{P}^N).
\end{array}
$$

Proof. In Proposition 4.1(c), $\gamma_{\mathcal{X}}(x) = \gamma_X(u')$ holds if and only if the linear subvariety $\gamma_X(u') \subset \mathbb{P}^N$ is of dimension m^+. Thus the assertion follows. □

In the case where π is generically finite, $\dim(X) = \dim(\mathcal{X}) + m$; hence we also have:

Corollary 4.4. *Assume that π is separable and generically finite. Then we have* $\dim(\mathcal{X}) = m^+ - m$ *if and only if $\mathcal{X}$ is developable.*

If the Gauss map of X is separable, then there exists the maximal developable parameter space $(\mathcal{X}_0, X)$, as follows:

Remark 4.5. For a projective variety $X \subset \mathbb{P}^N$ whose Gauss map γ_X is separable, we set

$$
\mathcal{X}_0 \subset \mathbb{G}(m_0, \mathbb{P}^N)
$$

to be the closure of the space which parametrizes (closures of) general fibers of γ_X, where the projection $\pi_0 : U_{\mathcal{X}_0} \to X$ is birational and the expanding map $\gamma_{\mathcal{X}_0}$ is birational. We can obtain $\mathcal{X}_0$ as the closure of the image of X under the composite map $\sigma_{\mathcal{Y}} \circ \gamma_X$, due to Corollary 3.2. Here, $(\mathcal{X}_0, X)$ is the *maximal* developable parameter space; in other words, for any developable $(\mathcal{X}, X)$ with $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$, there exists a dominant rational map $\mathcal{X} \dashrightarrow \mathcal{X}_0$ through which $\gamma_{\mathcal{X}} : \mathcal{X} \dashrightarrow \mathcal{Y}$ factors. (This is because, for each $x \in \mathcal{X}$, we have an inclusion $x \cap X^{sm} \subset \gamma_X^{-1}(y)$ in $\mathbb{P}^N$ with $y := \gamma_{\mathcal{X}}(x)$. Indeed, since $\gamma_{\mathcal{X}_0} \circ \sigma_{\mathcal{Y}} = id$, the map $\mathcal{X} \dashrightarrow \mathcal{X}_0$ is given by $\sigma_{\mathcal{Y}} \circ \gamma_{\mathcal{X}}$.)

Remark 4.6. Let $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$ be a subvariety such that $(\mathcal{X}, X)$ is developable.

(a) Assume that π is separable and generically finite, and assume that $\gamma_{\mathcal{X}}$ is generically finite. Then π is indeed birational (i.e, X is separably ruled by m-planes). The reason is as follows: From the diagram (19), for general $x \in \mathcal{X}$, since m is equal to the dimension of the fiber of $\gamma_X^{-1}(\gamma_{\mathcal{X}}(x))$, the m-plane $x \subset \mathbb{P}^N$ is set-theoretically equal to an irreducible component of the closure of the fiber $\gamma_X^{-1}(\gamma_{\mathcal{X}}(x))$. This implies that π is generically injective, and hence is birational.

(b) Assume that π and $\gamma_{\mathcal{X}}$ are separable and generically finite. Then $\mathcal{X}$ is equal to the parameter space $\mathcal{X}_0$ given in Remark 4.5. The reason is as follows: If $\gamma_{\mathcal{X}}$ is separable, then so is γ_X. It follows from Corollary 3.6 that the closure of the fiber $\gamma_X^{-1}(\gamma_{\mathcal{X}}(x))$ is irreducible, and hence is equal to the m-plane x. Thus $\mathcal{X} = \mathcal{X}_0$.

For example, in the following situation, the maximal developable parameter space for the dual variety of $X \subset \mathbb{P}^N$ can be obtained:

Proposition 4.7. *Let $\gamma_X : X \dashrightarrow \mathcal{Y} \subset \mathbb{G}(\dim(X), \mathbb{P}^N)$ be the Gauss map, and let $Y := \bar{\pi}(V_{\mathcal{Y}})$ in $(\mathbb{P}^N)^\vee$, the dual variety of X. If X is reflexive and $\bar{\pi}$ is generically finite, then $(\mathcal{Y}, Y)$ is the maximal developable parameter space with the birational projection $\bar{\pi} : V_{\mathcal{Y}} \to Y$, and then the following diagram is commutative:*

$$
\begin{array}{ccccc}
\gamma^* V_{\mathcal{Y}} & \dashrightarrow & V_{\mathcal{Y}} & \xrightarrow{\ \bar{\pi}\ } & Y \\
\downarrow & & \downarrow & & \Big\downarrow {\scriptstyle \sigma_Y} \\
X & \xdashrightarrow{\ \gamma_X\ } & \mathcal{Y} & \xdashrightarrow{\ \sigma_{\mathcal{Y}}\ } & \mathcal{X}_0,
\end{array}
$$

where note that the shrinking map $\sigma_Y : Y \dashrightarrow \mathcal{X}_0$ is identified with the Gauss map $\gamma_{Y/(\mathbb{P}^N)^\vee} : Y \dashrightarrow \mathbb{G}(N - m_0 - 1, (\mathbb{P}^N)^\vee)$.

Proof. Since X is reflexive, $\gamma^* \bar{\pi}$ is separable due to the Monge-Segre-Wallace criterion, and so is $\bar{\pi}$. Let $M := \dim X$. Since $\dim \mathcal{Y} = M - m_0 = (N - m_0 - 1) - (N - M - 1)$, it follows from Corollary 4.4 that $(\mathcal{Y}, Y)$ is developable and that the diagram is commutative. Since γ_X is separable, $\sigma_{\mathcal{Y}}$ is birational (see Corollary 3.2). Hence, considering the dual statement of Remark 4.6, we have the assertion. $\qquad \square$

Remark 4.8. Suppose that $\mathcal{Y}$ is of dimension one. Then $\bar{\pi}$ is always separable and generically finite (see Lemma 4.9 below). In this case, X is reflexive if and only if γ_X is separable.

4.2. **One-dimensional developable parameter space.** In this subsection, we assume that $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$ is a projective curve. As above, we denote by $\pi = \pi_{\mathcal{X}} : U_{\mathcal{X}} \to \mathbb{P}^N$ the projection, and by $X := \pi(U_{\mathcal{X}})$ in $\mathbb{P}^N$. Here separability of π always holds; this is deduced from [14], and can be also shown, as follows:

Lemma 4.9. *Let $\mathcal{X}$ be as above. Then π is separable and generically finite.*

Proof. Note that $U_{\mathcal{X}}$ is of dimension $m + 1$. Since $\mathcal{X}$ is a curve, it follows from Proposition 4.1(b) that $\operatorname{rk} d_u \pi \geqslant m + 1$. Thus $\operatorname{rk} d_u \pi = m + 1$, which implies that π is separable and generically finite. $\qquad \square$

Let us consider the expanding map $\gamma : \mathcal{X} \dashrightarrow \mathbb{G}(m_\gamma^+, \mathbb{P}^N)$ and shrinking map $\sigma : \mathcal{X} \dashrightarrow \mathbb{G}(m_\sigma^-, \mathbb{P}^N)$ of $\mathcal{X}$, with integers m_γ^+ and m_σ^- ($m_\sigma^- < m < m_\gamma^+$).

Lemma 4.10. $m_\gamma^+ + m_\sigma^- = 2m$.

Proof. In the setting of §2.2, we consider the matrix

$$
F = \begin{bmatrix} f_{0,z}^{m+1} & \cdots & f_{0,z}^N \\ \vdots & & \vdots \\ f_{m,z}^{m+1} & \cdots & f_{m,z}^N \end{bmatrix},
$$

where note that, since $\mathcal{X}$ is of dimension one, the system of parameters of $\mathcal{O}_{\mathcal{X},x}$ consists of one element z. Recall that we have $\operatorname{rk} F = \dim(\varphi_x(S)) = m_\gamma^+ - m$. In the same way, we have $\operatorname{rk} F = m - m_\sigma^-$. Thus the assertion follows. $\qquad\square$

Corollary 4.11. *The following are equivalent:*

(a) $\mathcal{X}$ *is developable.*
(b) $m_\gamma^+ = m + 1$.
(c) $m_\sigma^- = m - 1$.

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from Corollary 4.4. The equivalent (b) $\Leftrightarrow$ (c) follows from Lemma 4.10. $\qquad\square$

Recall that $\gamma^*\bar{\pi}$ is the generalized conormal morphism given in Definition 2.7.

Lemma 4.12. *Assume that γ is generically finite. Then, γ is separable if and only if so is $\gamma^*\bar{\pi} : \gamma^*V_{\mathbb{G}(m_\gamma^+,\mathbb{P}^N)} \to (\mathbb{P}^N)^\vee$.*

Proof. For $\mathcal{Y} \subset \mathbb{G}(m_\gamma^+,\mathbb{P}^N)$, the closure of the image of γ, we set $Y \subset (\mathbb{P}^N)^\vee$ to be the image of $V_{\mathcal{Y}}$ under $\bar{\pi}$. Since $\mathcal{Y}$ is of dimension one, $\bar{\pi}$ is separable and generically finite, due to Lemma 4.9. Hence the assertion follows. $\qquad\square$

Considering the dual of the above statement, we also have:

Corollary 4.13. *Assume that σ is generically finite. Then σ is separable if and only if so is $\sigma^*\pi : \sigma^*U_{\mathbb{G}(m_\sigma^-,\mathbb{P}^N)} \to \mathbb{P}^N$.*

Remark 4.14. (a) If $(\mathcal{X}, X)$ is developable and γ is generically finite, then π is birational, due to Lemma 4.9 and Remark 4.6. Moreover, if γ is separable, then we have $\mathcal{X} = \mathcal{X}_0$.

(b) If $X \subset \mathbb{P}^N$ is non-degenerate and is not a cone, then it follows from Lemma 2.6 that γ and σ are generically finite.

Recall that γ^i and σ^i are composite maps given in §1.3. We denote by $TX = T^1X := \overline{\bigcup_{x\in X^{sm}} \mathbb{T}_x X} \subset \mathbb{P}^N$, the *tangent variety*, and by $T^0X := X$, $T^iX := T(T^{i-1}X)$.

Theorem 4.15. *Let $\mathcal{X} \subset \mathbb{G}(m,\mathbb{P}^N)$ and $\mathcal{X}' \subset \mathbb{G}(m',\mathbb{P}^N)$ be projective curves with projections $\pi_{\mathcal{X}} : U_{\mathcal{X}} \to X$ and $\pi_{\mathcal{X}'} : U_{\mathcal{X}'} \to X'$. Then, for an integer $\varepsilon \geqslant 0$, the following are equivalent:*

(a) $(\mathcal{X}', X')$ *is developable, $\gamma^\varepsilon = \gamma_{\mathcal{X}'}^\varepsilon$ is separable, $\gamma^\varepsilon\mathcal{X}' = \mathcal{X}$, and X' is non-degenerate and is not a cone.*
(b) $(\mathcal{X}, X)$ *is developable, $\sigma^\varepsilon = \sigma_{\mathcal{X}}^\varepsilon$ is separable, $\sigma^\varepsilon\mathcal{X} = \mathcal{X}'$, and X is non-degenerate and is not a cone.*

In this case, $m = m' + \varepsilon$ and $X = T^\varepsilon X'$.

Proof. (b) $\Rightarrow$ (a): It is sufficient to show the case $\varepsilon = 1$. Since $\mathcal{X}$ is developable, it follows from Corollary 4.11 that $m' = m_\sigma^-$ is equal to $m-1$. From Lemma 2.6, $\sigma_{\mathcal{X}}$ is generically finite. From Corollary 4.13, $\sigma_{\mathcal{X}}^*\pi$ is separable. Applying Corollary 3.2, we have that $\sigma_{\mathcal{X}} \circ \gamma_{\mathcal{X}'}$ gives an identity map of an open subset

of $\mathcal{X}'$, and that $\mathcal{X}'$ is developable. In addition, $\mathcal{X}$ is equal to the image of the Gauss map $\gamma_{X'}$; hence the image of $U_{\mathcal{X}} \to \mathbb{P}^N$ is equal to TX', which means that $X = TX'$.

The converse (a) $\Rightarrow$ (b) follows in the same way. $\qquad\square$

In the statement of (a) of Theorem 4.15, if $m' = 0$ and $C := \mathcal{X}' \subset \mathbb{P}^N$, then we regard C itself as a developable parameter space (of 0-planes). Note that, the osculating scroll of order i of a curve $C \subset \mathbb{P}^N$ coincides with $T^i C$ if the characteristic is zero or satisfies some conditions (Homma [9, §2]).

Corollary 4.16. *Assume one of the conditions* (a) *and* (b) *of Theorem 4.15, and assume that $m' = 0$, i.e., $C := \mathcal{X}'$ is a curve in $\mathbb{P}^N$. Then $C = \sigma^m \mathcal{X}$ and $\mathcal{X} = \gamma^m C$ holds. In particular, we have $X = T^m C$, where $T^i C$ coincides with the osculating scroll of order i of C for each $0 < i \leqslant m + 1$. In addition, if $\gamma_{\mathcal{X}}$ is separable (equivalently, γ_X is separable) and $m + 1 < N$, then $\mathcal{X}$ is the closure of the space which parametrizes general fibers of γ_X.*

In the case where $X \subset \mathbb{P}^N$ is a cone with maximal vertex L, considering the linear projection from L and using Corollary 4.16, we have that X is a cone over an osculating scroll of order $m - \dim(L) - 1$ of a certain curve in $\mathbb{P}^{N - \dim(L) - 1}$ if $(\mathcal{X}, X)$ is developable and $\sigma^{m - \dim(L) - 1}$ is separable.

Proof of Corollary 4.16. The former statement follows from Theorem 4.15; in particular, γ^m is separable, $\mathcal{X} = \gamma^m C$, and $X = T^m C$. From the diagram (19) of Corollary 4.3, for $0 \leqslant i < m$, $\gamma_{T^i C} : T^i C \dashrightarrow \gamma^{i+1} C$ is separable; then $T^i C$ is reflexive as in Remark 4.8. By induction, it follows from [9, Corollary 2.3 and Theorem 3.3] that $T^{i+2} C$ coincides with the osculating scroll of order $i + 2$ of C. On the other hand, if $\gamma_{\mathcal{X}}$ is separable, then $\mathcal{X} = \mathcal{X}_0$ as in Remark 4.14. $\quad\square$

Let us consider $X^* \subset (\mathbb{P}^N)^\vee$, the dual variety of $X \subset \mathbb{P}^N$. Then we have the following relation with dual varieties and tangent varieties.

Corollary 4.17. *Let m, ε be integers with $\varepsilon > 0$ and $m + \varepsilon + 1 < N$, let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety of dimension $m + 1$, and let $\mathcal{X} \subset \mathbb{G}(m, \mathbb{P}^N)$ be a projective curve such that $(\mathcal{X}, X)$ is developable. If $\gamma^{\varepsilon+1} = \gamma_{\mathcal{X}}^{\varepsilon+1}$ is separable, then we have $T^\varepsilon((T^\varepsilon X)^*) = X^*$ in $(\mathbb{P}^N)^\vee$.*

Proof. We can assume that X is not a cone. By definition, X is the image of $U_{\mathcal{X}} \to \mathbb{P}^N$. From Theorem 4.15, $T^\varepsilon X$ is given by the image of $U_{\gamma^\varepsilon \mathcal{X}} \to \mathbb{P}^N$. Let $\mathcal{Y} := \gamma^{\varepsilon+1} \mathcal{X}$, and let Y be the image of $V_{\mathcal{Y}} \to (\mathbb{P}^N)^\vee$. Since $\sigma^\varepsilon = \sigma_{\mathcal{Y}}^\varepsilon$ is separable, considering the dual of the above statement, we have that $T^\varepsilon Y$ is given by the image of $V_{\sigma^\varepsilon \mathcal{Y}} \to (\mathbb{P}^N)^\vee$. On the other hand, for each $0 \leqslant i \leqslant \varepsilon$, from the diagram (19), since $\gamma^{i+1} \mathcal{X}$ is equal to the image of the Gauss map $\gamma_{T^i X}$, the dual variety $(T^i X)^*$ is given by the image of $V_{\gamma^{i+1} \mathcal{X}} \to (\mathbb{P}^N)^\vee$ (see Definition 2.7, Proposition 4.7). In particular, $Y = (T^\varepsilon X)^*$. From Theorem 4.15, it follows that $\sigma^\varepsilon \mathcal{Y} = \gamma^1 \mathcal{X}$. Hence $T^\varepsilon(Y)$ and X^* coincide, since these are given by the image of $V_{\gamma^1 \mathcal{X}} \to (\mathbb{P}^N)^\vee$. $\qquad\square$

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